Section 2.2 Notes

The Algebra of Functions

1 Combining Functions Algebraically

Adding two Functions

- If f and g are two functions,
$$(f+g)(x) = f(x) + g(x)$$
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x - 2$:

$$(f+g)(x) = \sqrt{x+1} + x - 2$$

Subtracting two Functions

- If f and g are two functions,
$$(f+g)(x) = f(x) - g(x)$$
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x - 2$:

$$(f-g)(x) = \sqrt{x+1} - (x-2) = \sqrt{x+1} - x + 2$$

Multiplying two Functions

- If f and g are two functions,
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x - 2$:

$$(f \cdot g)(x) = \sqrt{x+1} \cdot (x-2) = (x-2)\sqrt{x+1}$$

Dividing two Functions

- If f and g are two functions,
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x - 2$:

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+1}}{x-2}$$

Finding the Domain of the New Function

- The domain of $f+g$, $f-g$, and $f \cdot g$ is just the overlap of the domains of f and g.
- The domain of $\frac{f}{g}$ is the overlap of the domains of f and g, but with any points where $g(x) = 0$ removed.
- To find the domain algebraically:
    1. Look at the *unsimplified* formula.
    2. Apply the usual rules for finding domains (division, square roots.)
    3. You might get several inequalities you need to solve. Treat these like the "and"-type of compound inequalities to get the final answer.

Rules for Finding Domain

1. If the expression contains $\boxed{\dfrac{A}{B}}$, then $\boxed{B \neq 0}$.
2. If the expression contains $\boxed{\sqrt{B}}$, then $\boxed{B \geq 0}$.

- If the expression contains $\boxed{\dfrac{A}{\sqrt{B}}}$, then $\boxed{B > 0}$. This rule *only* works when the *entire* denominator is under a square root.

Examples

Find the new functions and determine their domains.

1. $f(x) = \frac{1}{x-2}$, $g(x) = \sqrt{3x-1}$; find $(f+g)(x)$

$$(f+g)(x) = \frac{1}{x-2} + \sqrt{3x-1}$$

Domain: $\left[\frac{1}{3}, 2\right) \cup (2, \infty)$

2. $f(x) = \sqrt{x}$, $g(x) = \sqrt{2-x}$; find $(f \cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$

$$(f \cdot g)(x) = \sqrt{x(2-x)}$$

Domain: $[0, 2]$

$$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x}{2-x}}$$

Domain: $[0, 2)$

3. $f(x) = x^2 - 1$, $g(x) = x - 1$; find $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = x + 1$$

Domain: $(-\infty, 1) \cup (1, \infty)$

Evaluate each function at the point, if it exists.

4. $f(x) = \sqrt{3+x}$, $g(x) = \sqrt{2x+10}$, find $(f \cdot g)(-1)$.

    $\boxed{4}$

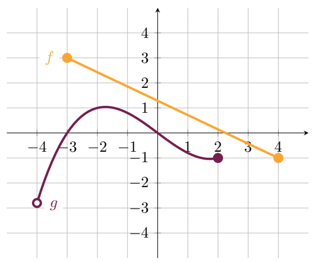
5. $f(x) = 3x+1$, $g(x) = 4-x^2$, find $\left(\frac{f}{g}\right)(2)$.

    Does not exist.

6. $f(x) = \sqrt{3-x}$, $g(x) = 4-x$, find $(f+g)(4)$.

    Does not exist.

7. Find the domain of $\frac{f}{g}$, where f and g are the functions graphed below:



    $\boxed{(-3,0) \cup (0,2]}$

2 Difference Quotient

Definition
For a function $f(x)$, the difference quotient is given by the formula

$$\boxed{\frac{f(a+h)-f(a)}{h}}$$

The goal for these problems is basically to simplify until you can cancel out the h in the denominator.

Examples
Compute the difference quotient for each function.

1. $f(x) = x^2 + 1$

    $\boxed{2a+h}$

2. $g(x) = \frac{1}{x+3}$

    $\boxed{-\frac{1}{(a+h+3)(a+3)}}$

3. $f(x) = x^3$

    $\boxed{3a^2+3ah+h^2}$