Section 3.1 Notes

The Complex Numbers

1 Introduction

Complex Numbers

- In this section, we break one of our rules: don't take the square root of a negative number!
- To do this, mathematicians made up a new number called i:
  - $\mathrm{i} = \sqrt{-1}$
  - The essential property of i is that $\mathrm{i}^2 = -1$.
  - i is a number, but it is not a *Real Number* - it is sometimes called the "imaginary unit."
- A *complex number* has the form $\boxed{a + b\mathrm{i}}$, where a and b are real numbers.
  - a is called the *real part*
  - b is called the *imaginary part*

2 Arithmetic Operations

Addition and Subtraction

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$
$$(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}$$

- Add or subtract like terms
- Example 1: $(2 + \mathrm{i}) + (4 - 3\mathrm{i}) = 6 - 2\mathrm{i}$
- Example 2: $(4 - 2\mathrm{i}) - (4 - 8\mathrm{i}) = 4 - 2\mathrm{i} - 4 + 8\mathrm{i} = 6\mathrm{i}$

Multiplication

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

- FOIL, and use the property that $\mathrm{i}^2 = -1$ to fully simplify.
- For example:

$$\begin{aligned}(2 - 3\mathrm{i})(1 + 4\mathrm{i}) &= 2 + 8\mathrm{i} - 3\mathrm{i} - 12\mathrm{i}^2 \\ &= 2 + 5\mathrm{i} - 12(-1) \\ &= 2 + 12 + 5\mathrm{i} \\ &= 14 + 5\mathrm{i}\end{aligned}$$

Division

$$\frac{a+b\mathrm{i}}{c+d\mathrm{i}} = \frac{ac+ad}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}$$

- When you are simplifying $\dfrac{a+b\mathrm{i}}{c+d\mathrm{i}}$, the goal is to get it back into the form of a normal complex number: $\boxed{} + \boxed{}\mathrm{i}$.
- Multiply the top and bottom by the conjugate of the bottom and simplify:

$$c+d\mathrm{i} \longleftrightarrow c-d\mathrm{i}$$

- Notation: If z is a complex number, its conjugate is often indicated by $\overline{z}$. In other words, if $z = a+b\mathrm{i}$, then $\overline{z} = a - b\mathrm{i}$

Examples

1. $\dfrac{2-3\mathrm{i}}{4+3\mathrm{i}}$

   $$\boxed{\frac{5}{19} - \frac{18}{19}\mathrm{i}}$$

2. $\dfrac{3\mathrm{i}}{2-7\mathrm{i}}$

   $$\boxed{-\frac{21}{53} + \frac{6}{53}\mathrm{i}}$$

3. $\dfrac{3+\mathrm{i}}{15\mathrm{i}}$

   $$\boxed{\frac{1}{15} - \frac{1}{5}\mathrm{i}}$$

3 Square Roots of Negative Numbers

Simplifying Square Roots with Negative Numbers

- Unfortunately, if a and b are *both* negative,

$$\sqrt{ab} \neq \sqrt{a}\sqrt{b}$$

- However, as long as just *one* of a or b is nonnegative, the property does hold:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

- Specifically, we can pull out $\sqrt{-1}$:

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2\mathrm{i}$$

Examples
Simplify.

1. $\sqrt{\dfrac{1}{4}}\sqrt{-64}$

   $\boxed{4\mathrm{i}}$

2. $\dfrac{\sqrt{-36}}{\sqrt{-4}\sqrt{-9}}$

   $\boxed{-\mathrm{i}}$

4 Powers of i

Computing Large Powers of i

- When we raise i to different exponents, a pattern emerges:
  - $\mathrm{i}^1 = \boxed{\mathrm{i}}$
  - $\mathrm{i}^2 = \boxed{-1}$
  - $\mathrm{i}^3 = \mathrm{i}^2 \cdot \mathrm{i} = -1 \cdot \mathrm{i} = \boxed{-\mathrm{i}}$
  - $\mathrm{i}^4 = \mathrm{i}^3 \cdot \mathrm{i} = -\mathrm{i} \cdot \mathrm{i} = -\mathrm{i}^2 = -(-1) = \boxed{1}$
  - $\mathrm{i}^5 = \mathrm{i}^4 \cdot \mathrm{i} = 1 \cdot \mathrm{i} = \boxed{\mathrm{i}}$
  - $\mathrm{i}^6 = \mathrm{i}^5 \cdot \mathrm{i} = \mathrm{i} \cdot \mathrm{i} = \mathrm{i}^2 = \boxed{-1}$
  - $\mathrm{i}^7 = \mathrm{i}^6 \cdot \mathrm{i} = -1 \cdot \mathrm{i} = \boxed{-\mathrm{i}}$
  - $\mathrm{i}^8 = \mathrm{i}^7 \cdot \mathrm{i} = -\mathrm{i} \cdot \mathrm{i} = -\mathrm{i}^2 = -(-1) = \boxed{1}$
- The powers keep cycling through i, -1, $-\mathrm{i}$, and 1.
- Notice how if the exponent is a multiple of 4, then the answer is 1.

Examples
Simplify.

1. i^{107}

   $\boxed{-\mathrm{i}}$

2. i^{30}

   $\boxed{-1}$