Section 3.2 Notes

Quadratic Equations, Functions, Zeros, and Models

1 Quadratic Equations

Definition

- A *quadratic equation* is one that can be simplified to the form $ax^2 + bx + c = 0$.
- A *quadratic function* has the form $f(x) = ax^2 + bx + c$.
- x is the variable - but other variables are fine too
- a, b and c are numbers, but $a \neq 0$

Quadratics Missing bx

- When the equations is missing the middle term, it can always be simplified to look like

$$x^2 = a$$

- To solve, simply take the square root on both sides (to cancel out the square), and remember to insert the $+/-$

Example
Solve the quadratic equation.

$$3x^2 = 24$$

$$\boxed{x = \pm 2\sqrt{2}}$$

Factoring
If the equation can be rewritten as $(ax+b)(cx+d) = 0$, then the equations $ax+b = 0$ and $cx+d = 0$ give the solutions to the equation.

- Be aware that not every quadratic can be factored, and so this method will often fail. However, you can *always* use the quadratic formula.

Example
Solve the quadratic equation.

$$10x^2 + x - 3 = 0$$

$$\boxed{x = -\frac{3}{5},\ x = \frac{1}{2}}$$

Quadratic Formula
The equation $ax^2 + bx + c = 0$ can be solved with the following formula:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The portion under the root $(b^2 - 4ac)$ is called the *discriminant*:

- If the discriminant is negative, there are two complex solutions.
- If the discriminant is zero, there is exactly on real solution.
- If the discriminant is positive, there are exactly two real solutions.

Example

1. Solve the quadratic equation: $2x^2 - 3x = 5$

   $\boxed{x = \frac{5}{2},\ x = -1}$

2. Find the zeros of the function: $f(x) = \frac{1}{2}x^2 - x + 5$

   $\boxed{1 \pm 3\text{i}}$

Completing the Square
Let's illustrate the steps with an example: $4x^2 + 16x - 9 = 0$

1. Move the constant term to the other side.

$$4x^2 + 16x = 9$$

2. Divide everything by the coefficient of x^2.

$$x^2 + 4x = \frac{9}{4}$$

3. The equation should look like $x^2 + Bx = C$. Do some scratch work to compute:
   (a) $\frac{B}{2}$
   (b) $\left(\frac{B}{2}\right)^2$

$$\begin{aligned} &(a)\ \frac{4}{2} = 2 \\ &(b)\ 2^2 = 4 \end{aligned}$$

4. Add $\left(\frac{B}{2}\right)^2$ to both sides of the equation.

$$\begin{aligned} x^2 + 4x + 4 &= \frac{9}{4} + 4 \\ x^2 + 4x + 4 &= \frac{9}{4} + \frac{16}{4} \\ x^2 + 4x + 4 &= \frac{25}{4} \end{aligned}$$

5. The left-hand-side (LHS) of the equation will now *always* factor into $\left(x+\frac{B}{2}\right)^2$.

$$(x+2)^2 = \frac{25}{4}$$

6. Solve using the final equation solving technique: "If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$."

$$\begin{aligned} x+2 &= \pm\sqrt{\frac{25}{4}} \\ x+2 &= \pm\frac{5}{2} \\ x &= -2 \pm \frac{5}{2} = -\frac{4}{2} \pm \frac{5}{2} = \frac{-4\pm 5}{2} \end{aligned}$$

$x = \frac{-4+5}{2} = \boxed{\frac{1}{2}}$ or $x = \frac{-4-5}{2} = \boxed{-\frac{9}{2}}$.

Examples
Solve the quadratic equation.

1. $4x^2 - 5x + 1 = 0$

   $\boxed{x = 1 \text{ or } x = \frac{1}{4}}$

2. $3x^2 + 8x + 1 = 0$

   $\boxed{x = \frac{-4\pm\sqrt{13}}{3}}$

3. $ax^2 + bx + c = 0$

   $\boxed{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

2 Quadratic-Type Equations

Definition
A *quadratic-type* equation has three properties:

1. Three terms
2. Two terms have variables - the third should not
3. The exponent on one of the variables is twice the other.

Solving using Substitution
Let's illustrate with $8x^6 - 17x^3 + 9 = 0$.

1. Figure out which exponent is twice the other. Rewrite as a square.

$$8(\boxed{x^3})^2 - 17\boxed{x^3} + 9 = 0$$

2. Make a substitution for the repeated expression. You can use any variable that's not in the original problem.

$$u = \boxed{x^3}$$
$$8u^2 - 17u + 9 = 0$$

3. Solve for the new variable.

$$(8u - 9)(u - 1) = 0$$
$$8u - 9 = 0 \text{ or } u - 1 = 0$$
$$u = \frac{9}{8} \text{ or } u = 1$$

4. Replace the original variable and solve.

$$x^3 = \frac{9}{8} \text{ or } x^3 = 1$$
$$x = \sqrt[3]{\frac{9}{8}} \text{ or } x = \sqrt[3]{1}$$
$$x = \frac{\sqrt[3]{9}}{2} \text{ or } x = 1$$

Example
Solve for y:

1. $(2x + 3)^2 - (2x + 3) - 30 = 0$

$$\boxed{x = -4 \text{ or } x = \tfrac{3}{2}}$$

2. $4y^4 - 4y^2 - 3 = 0$

$$\boxed{x = \pm\sqrt{\tfrac{3}{2}},\ x = \pm\tfrac{i}{\sqrt{2}}}$$