Section 4.1 Notes

Polynomial Functions and Models

1 Introduction

Definitions

- A *polynomial function* is a function with the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where

  - a_n, $a_{n-1}, \ldots, a_1$, and a_0 are numbers, and are known as *coefficients*
  - a_n is the *leading coefficient* (and should be $\neq 0$)
  - n is the *degree*
  - $a_n x^n$ is the *leading term*
  - a_0 is the *constant term*

- For example, $P(x) = 3x^2 - 2$ and $P(x) = x^5 + 4x^4 - x$ are polynomial functions.

Types of Polynomials

- If the function has degree 0, i.e. $f(x) = a$, the function is called *constant*.
- If the function has degree 1, i.e. $f(x) = ax + b$, the function is called *linear*.
- If the function has degree 2, i.e. $f(x) = ax^2 + bx + c$, the function is called *quadratic*.
- If the function has degree 3, i.e. $f(x) = ax^3 + bx^2 + cx + d$, the function is called *cubic*.
- If the function has degree 4, i.e. $f(x) = ax^4 + bx^3 + cx + d + e$, the function is called *quartic*.

Examples

- Identify the leading term, leading coefficient, degree, and classify the polynomial as constant, linear, quadratic, cubic, or quartic for

$$f(x) = -5x^4 + 2x^2$$

Leading Term: $-5x^4$	Degree: 4
Leading Coefficient: -5	Type of Polynomial: quartic

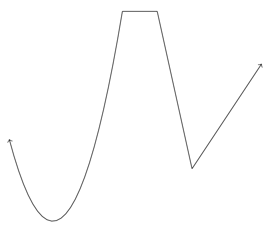
- Find the leading term, leading coefficient, and degree of the polynomial:

$$f(x) = 5(2x^3 + 1)^2(-x^2 + 4)$$

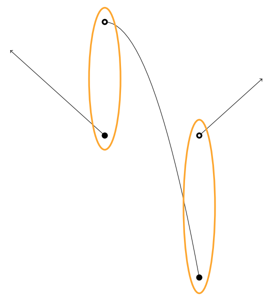
Leading Term: $-20x^8$	Degree: 8
Leading Coefficient: -20	

2 Properties of Polynomial Graphs

Continuity
The graphs of polynomial functions are *continuous* : there are no breaks in the graph, and can be drawn without lifting the pen or pencil.

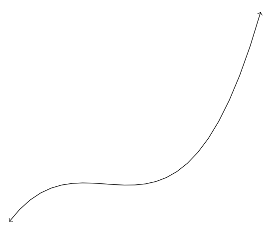


This is continuous.

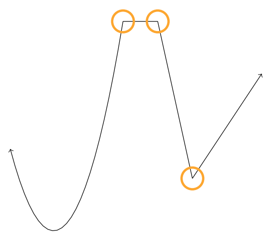


This is *not* continuous.

Smoothness
The graphs of polynomial functions are *smooth* : there are no sharp corners/cusps.



This is smooth.

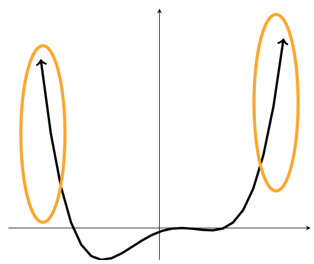


This is not smooth.

3 End Behavior

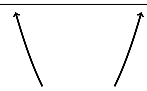
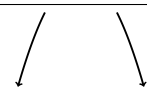
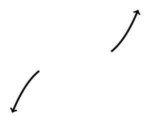
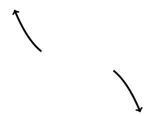
Definition

- The *end behavior* of a polynomial is a description of what happens to the y-values as you plug in extremely large (positive or negative) x-values.
- In terms of the graph, it's useful to think of the end behavior as the the appearance of the graph on the two sides outside of all the x-intercepts.



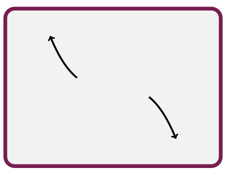
Determining the End Behavior
To figure out the end behavior, the only thing that matters is the leading term.

Suppose $\mathbf{ax^n}$ is the leading term:

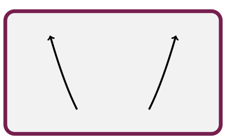
	$a > 0$	$a < 0$
n is even	 As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to \infty$	 As $x \to -\infty$, $y \to -\infty$ As $x \to \infty$, $y \to -\infty$
n is odd	 As $x \to -\infty$, $y \to -\infty$ As $x \to \infty$, $y \to \infty$	 As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to -\infty$

Examples
Determine the end behavior of the polynomial.

1. $P(x) = -2x^5 + x^2 - 1$



2. $P(x) = 3x^4 - x^3 + x^2 + 5x + 8$



4 Multiplicity of Zeros

Zeros of a Polynomial

- Recall that *zeros* (or *roots*, *x-intercepts*) are points where the graph crosses the x-axis, and are the values of x when $y = 0$.
- To find the zeros of a polynomial, you must use the factored form of the polynomial.
- Plug in 0 for $f(x)$, then set each factor equal to zero and solve.

  For example, we can find the zeros of $P(x) = x^3 - 3x^2 - 4x + 12$:

  $0 = (x^3 - 3x^2) + (-4x + 12)$

  $0 = x^2(x - 3) - 4(x - 3)$

  $0 = (x - 3)(x^2 - 4)$

  $0 = (x - 3)(x - 2)(x + 2)$

$x - 3 = 0$, $x - 2 = 0$, $x + 2 = 0$

$x = 3$, $x = 2$, $x = -2$

Definition of Multiplicity

- The *multiplicity* of a zero a is the number of linear factors that, when set equal to zero, simplify to $x = a$.

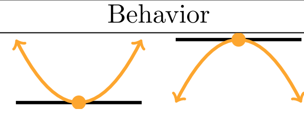
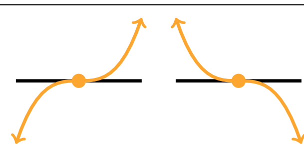
  For example, if $P(x) = (x - 5)(x - 5)(x + 2)$, then $x = 5$ has a multiplicity of 2 and $x = -2$ has a multiplicity of 1.

- When you have a repeated factor, normally you'll see it written more simply using exponents. In this case, the multiplicity is just the exponent of the factor that gives $x = a$.

  For example, if $P(x) = (x + 10)^4(x - 6)^3(x + 1)$, then $x = -10$ has a multiplicity of 4, $x = 6$ has a multiplicity of 3, and $x = -1$ has a multiplicity of 1.

Behavior Near Zeros

The multiplicity of a zero determines the behavior of the graph at that zero:

Multiplicity	Behavior
even	 The graph bounces off the x-axis at the zero.
odd	 The graph crosses through the x-axis at the zero.

- The book refers to "bouncing" as being tangent to the x-axis.

Examples

1. Determine if 4 is a zero of $f(x) = x^3 - 2x^2 - 5x - 12$

   Yes.

2. Find the zeros and their multiplicities of $f(x) = x^4 - 4x^3 + 3x^2$

   0 with multiplicity 2
   1 with multiplicity 1
   3 with multiplicity 1

3. Find the zeros and their multiplicities of $g(x) = -3(4x^2 - 1)^4(-x + 4)^5$

   $\frac{1}{2}$ with multiplicity 4
   $-\frac{1}{2}$ with multiplicity 4
   4 with multiplicity 5

5 Modeling

Example

If there are x teams in a sports league and all the teams play each other twice, a total of $N(x)$ games are played, where

$$N(x) = x^2 - x$$

A softball league has 9 teams, each of which plays the other twice. If the league pays \$110 per game for the field and the umpires, how much will it cost to play the entire schedule?

$\boxed{\$7920}$