Section 4.3 Notes

Polynomial Division; The Remainder Theorem and the Factor Theorem

1 Polynomial Long Division

Example of Long Division of Numbers
Let's compute $\frac{823}{5}$:

$$\begin{array}{r}
164 \\
5\,\overline{)823} \\
\underline{-5} \\
32 \\
\underline{-30} \\
23 \\
\underline{-20} \\
3
\end{array}$$

The quotient is 164 and the remainder is 3:

$$\boxed{\frac{823}{5} = 164 + \frac{3}{5}}$$

Rules

- The polynomials must always be written in descending order of exponents.

  $(x^3 - 2x^4 + 5x^2) \to (-2x^4 + x^3 + 5x^2)$

- If you have any missing terms, fill those terms in using zeros for the coefficient:

  $(x^4 - 3x^2 + 1) \to (x^4 + 0x^3 - 3x^2 + 0x + 1)$

- If the coefficients don't divide in evenly, you'll end up with fractional coefficients - that's completely fine.

Example of Long Division with Polynomials
Let's compute $\frac{x^2-3x+4}{x+1}$:

$$\begin{array}{r}
x - 4 \\
x+1\,\overline{)\; x^2 - 3x + 4} \\
\underline{-\,x^2 \;\; -\,x} \\
-\,4x + 4 \\
\underline{4x + 4} \\
8
\end{array}$$

The quotient is $x - 4$ and the remainder is 8:

$$\boxed{\frac{x^2 - 3x + 4}{x + 1} = x - 4 + \frac{8}{x + 1}}$$

Examples

1. Divide $8x^4 + 6x^2 - 3x + 1$ by $2x^2 - x + 2$

   $$\boxed{4x^2 + 2x + \frac{-7x + 1}{2x^2 - x + 2}}$$

2. Compute $\dfrac{1+2x^2+3x^3}{2x-4}$

$$\boxed{\frac{3}{2}x^2+4x+8+\frac{33}{2x-4}}$$

2 Synthetic Division

Rules

- The polynomials must always be written in descending order of exponents.
- If you have any missing terms, fill those terms in using zeros for the coefficient.
- The divisor/denominator must have the form $x+c$ or $x-c$. If the divisor looks any different, you must go back to using long division.

Example of Synthetic Division

Let's compute $\frac{x^3-3x^2+1}{x-4}$:

$$\begin{array}{r|rrrr} & 1 & -3 & 0 & 1 \\ 4 & & 4 & 4 & 16 \\ \hline & \underbrace{1 \quad 1 \quad 4}_{\text{Quotient}} & & & \underset{\text{Remainder}}{\underset{\uparrow}{17}} \end{array}$$

The quotient is x^2+x+4 and the remainder is 17:

$$\boxed{\frac{x^2-3x+4}{x+1} = x-4+\frac{8}{x+1}}$$

Examples

- Divide x^3-27 by $x-3$.

  $$\boxed{x^2+3x+9}$$

- Compute $\frac{x^4+2x^2+x-4}{x+2}$.

  $$\boxed{x^3-2x^2+6x-11+\tfrac{18}{x+2}}$$

- Find the quotient and remainder of $(x^3-4\mathrm{i}x^2-x-3) \div (x+\mathrm{i})$.

  $$\boxed{\text{Quotient: } x^2-5\mathrm{i}x-6\text{; Remainder: } -3+6\mathrm{i}}$$

Forms of the Question

- So far, we've been asked to find $\frac{P(x)}{d(x)}$, which we've written as

$$\frac{P(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

- However, some questions will give you $P(x)$ and $d(x)$, and then ask you to find $Q(x)$ and $R(x)$ where

$$P(x) = d(x) \cdot Q(x) + R(x)$$

You solve this *exactly* the same way - they're just multiplying the first equation through by $d(x)$ to solve for $P(x)$.

Examples
For each $P(x)$ and $d(x)$, use long division or synthetic division to find $Q(x)$ and $R(x)$ so that

$$P(x) = d(x) \cdot Q(x) + R(x)$$

1. $P(x) = 4x^3 + 3x^2 - 2x + 5$ and $d(x) = x + 3$

   $$\boxed{4x^3 + 3x^2 - 2x + 5 = (x + 3) \cdot (4x^2 - 9x + 25) - 70}$$

2. $P(x) = x^4 - 2x^2 + x + 1$ and $d(x) = x^2 - x + 2$

   $$\boxed{x^4 - 2x^2 + x + 1 = (x^2 - x + 2) \cdot (x^2 + x - 3) - 4x + 7}$$

3 Factoring Polynomials

Identifying Factors by the Remainder

- When you divide $P(x)$ by $d(x)$, look what happens when the remainder is zero:

$$\begin{aligned} P(x) &= d(x) \cdot Q(x) + R(x) \\ P(x) &= d(x) \cdot Q(x) + 0 \\ P(x) &= d(x) \cdot Q(x) \end{aligned}$$

We've found a way to partially factor $P(x)$.

- Notice how this works the same as normal numbers:

$$\frac{20}{4} = 5 \text{ remainder } 0$$

so we know that 20 factors into $4 \cdot 5$.

Factoring using Synthetic Division
Let's factor $P(x) = x^3 + 3x^2 - 4$

1. You'll be making guesses using the follwing numbers:

$$1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

These correspond to the factors

$$(x - 1), (x + 1), (x - 2), (x + 2), (x - 3), (x + 3), (x - 4), (x + 4), \ldots$$

2. Go down the list, checking each possible factor using synthetic division until you get a remainder of 0.

$$\begin{array}{r|rrrr} & 1 & 3 & 0 & -4 \\ 1 & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array} \checkmark \qquad P(x) = (x-1)(x^2+4x+4)$$

3. Keep repeating the last step on just the "leftover" quotient until the "leftover" part is something you can factor - usually quadratic. Each time you successfully get a remainder of 0, try the same number again, because you can have repeated factors.

4. Once you get a quadratic (or something else you can factor), you can factor it normally.

$$P(x) = (x-1)(x^2+4x+4)$$
$$P(x) = (x-1)(x+2)(x+2)$$
$$P(x) = (x-1)(x+2)^2$$

Examples
Factor the polynomials.

1. $f(x) = x^4 - x^3 - 19x^2 + 49x - 30$

   $f(x) = (x-1)(x-2)(x+5)(x-3)$

2. $f(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4$

   $f(x) = (x+1)(x-2)^2\,(2x-1)$

4 Theorems

Remainder Theorem

If $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

- This comes from when you write the problem solved for $P(x)$.
- Remember that when you divide by $x - c$, your remainder is always just a number, which we'll call r:

$$P(x) = D(x) \cdot Q(x) + R(x)$$
$$P(x) = (x-c) \cdot Q(x) + r$$
$$P(c) = (c-c) \cdot Q(c) + r \text{ (Plug in } x = c.)$$
$$P(c) = 0 \cdot Q(c) + r$$
$$P(c) = r$$

Example

Compute $P(-7)$ using the Remainder Theorem.

$$P(x) = 5x^4 + 30x^3 - 40x^2 - 36x + 14$$

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Factor Theorem

c is a zero of $P(x)$ (in other words, $P(c) = 0$) exactly when $x - c$ is a factor of $P(x)$

- To see why this works, remember that c is a zero when $P(c) = 0$.
- If we use the Remainder Theorem, this means that the remainder when dividing by $x - c$ is 0.

$$\begin{aligned} P(x) &= D(x) \cdot Q(x) + R(x) \\ P(x) &= (x - c) \cdot Q(x) + 0 \\ P(x) &= (x - c) \cdot Q(x) \end{aligned}$$

Examples

1. Show that $c = 3$ is a zero of $P(x) = x^3 - x^2 - 11x + 15$ and find the other zeros.

   3 is a zero since $(x - 3)$ is a factor.
   The other zeros are $x = -1 \pm \sqrt{6}$

2. Is 2i a zero of the polynomial $x^4 + 3x^3 + 3x^2 + 12x - 4$?

   Yes, it is a zero.