Section 5.3 Notes

Logarithmic Functions and Graphs

1 Introduction

Definition

- $\log_a x$ answers the question "what exponent must a be raised to in order to get x?"
- In other words:
$$y = \log_a x \text{ means } a^y = x$$
- The *logarithmic function* with base a is given by the formula
$$f(x) = \log_a x$$
- Here a must be positive and $a \neq 1$.

Notation

- $\log_a x$ is also often written $\log_a(x)$.
- When written as $\log_a(x)$, this notation is meant to mirror $f(x)$ - function notation.
  - $\log_a$ is the name of the function
  - x is the input for the function
  - The parenthesis visually separate the name of the function from the input.
- When you speak about the function, it's okay to refer to "$\log_a$" by itself. However, when it's used in a formula or an equation, it must always have an input.
- For example
  - $y = 2 + \log_4(x)$ makes sense, whereas $y = 2 + \log_4$ does not.
  - To relate this back to function's we've dealt with before, $y = 2 + \sqrt[4]{x}$ makes sense, while $y = 2 + \sqrt[4]{}$ does not.

Examples

1. Convert to an exponential equation: $\log_5 625 = 4$

   $\boxed{5^4 = 625}$

2. Convert to a logarithmic equation: $4^5 = 1024$

   $\boxed{\log_4 1024 = 5}$

Calculate the following:

3. $\log_3 9$

   $\boxed{2}$

4. $\log_5 125$

   $\boxed{3}$

5. $\log_2 64$

   $\boxed{6}$

6. $\log_8 2$

   $\boxed{\frac{1}{3}}$

7. $\log_6 \frac{1}{36}$

   $\boxed{-2}$

2 Special Logarithms

Common Logarithm

$$\log x = \log_{10} x$$

This answers the question "what exponent must 10 be raised to in order to get x?"

Natural Logarithm

$$\ln x = \log_e x$$

This answers the question "what exponent must e (which is ≈ 2.7182818) be raised to in order to get x?"

Examples
Calculate the following:

1. $\log 1000$

   $\boxed{3}$

2. $\ln e^5$

   $\boxed{5}$

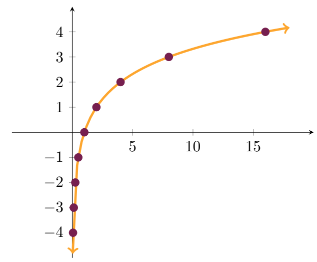
3. $\log \frac{1}{100}$

   $\boxed{-2}$

3 Logarithmic Graphs and Their Properties

Graphing Logarithmic Functions
Let's graph $y = \log_2 x$.

x	y
$1/16$	-4 (Since $2^{-4} = 1/2^4 = 1/16$)
$1/8$	-3 (Since $2^{-3} = 1/2^3 = 1/8$)
$1/4$	-2 (Since $2^{-2} = 1/2^2 = 1/4$)
$1/2$	-1 (Since $2^{-1} = 1/2^1 = 1/2$)
1	0 (Since $2^0 = 1$
2	1 (Since $2^1 = 2$)
4	2 (Since $2^2 = 4$)
8	3 (Since $2^3 = 8$)
16	4 (Since $2^4 = 16$)



Notice that the graph has a vertical asymptote along the y-axis.

Domain of Logarithmic Functions

In the last example, notice we didn't try to plug in $x = 0$ or any negative number.

Let's see what goes wrong:

$$y = \log_2(-8) \text{ means that } 2^y = -8$$

$\vdots$

$2^{-4} = \frac{1}{2^4} = \frac{1}{16} > 0$

$2^{-3} = \frac{1}{2^3} = \frac{1}{8} > 0$

$2^{-2} = \frac{1}{2^2} = \frac{1}{4} > 0$

$2^{-1} = \frac{1}{2^1} = \frac{1}{2} > 0$

$2^0 = 1 > 0$

$2^1 = 2 > 0$

$2^2 = 4 > 0$

$2^3 = 8 > 0$

$2^4 = 16 > 0$

$\vdots$

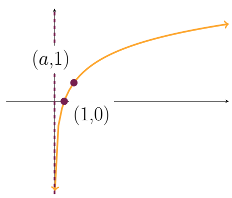
- No matter what we plug in for y, every answer ends up positive!
- We *can't* plug a negative number or zero into a logarithm.
- The domain of $y = \log_2 x$ is $(0, \infty)$.

Rules for Finding Domain

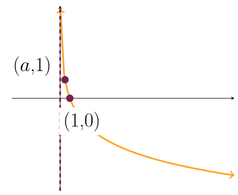
1. If the expression contains $\boxed{\dfrac{A}{B}}$, then $\boxed{B \neq 0}$.
2. If the expression contains $\boxed{\sqrt{B}}$, then $\boxed{B \geq 0}$.
3. If the expression contains $\boxed{\log_a B}$, then $\boxed{B > 0}$.

General Logarithmic Graph

In general, there are just two basic shapes for $f(x) = \log_a x$:



$a > 1$

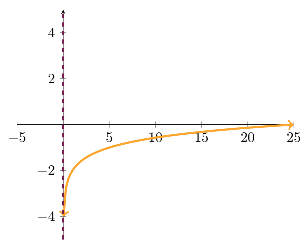


$a < 1$ (This is pretty uncommon)

- Again, they both have vertical asymptotes along the y-axis.
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$

Examples
Graph the following and find its domain:

$$y = \log_5(x) - 2$$

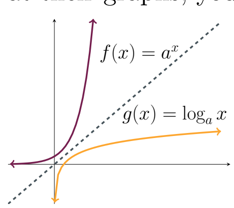


Domain: $(0, \infty)$

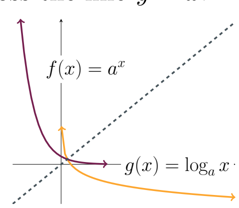
4 Relationship to Exponential Functions

Relationship through Graphs

- It turns out, $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions.
- If you look at their graphs, you can see they're reflections across the line $y = x$:



$a > 1$



$a < 1$

5 Change of Base Formula

Formula

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

- The purpose of this formula is to allow you to switch to any new base you want (you can pick any positive number for b except for 1).
- Switching can allow you to plug things into a calculator (most calculators only have ln and log buttons).

Examples

1. $\log_{11} 2$

   $\boxed{0.289064\ldots}$

2. $\log_5 37$

   $\boxed{2.243589\ldots}$

6 Applications

Example

In chemistry, the pH of a substance is defined as

$$\text{pH} = -\log[\text{H}^+]$$

where H^+ is the hydrogen ion concentration, in moles per liter. Hair conditioner has a pH of about 2.9. Find the hydrogen ion concentration.

$\boxed{0.00126 \text{ moles/liter}}$