Section 5.5 Notes

Solving Exponential Equations and Logarithmic Equations

1 Exponential Equations

Definition

- An *exponential equation* is one where the variable appears in an exponent.
- There are a few different categories that these equations can fall into.

The equation simplifies to $a^X = a^Y$

- Set $X = Y$ and finish solving.
- For example:

$$\begin{aligned} 5^x - 2 &= 23 \\ 5^x &= 25 \\ 5^x &= 5^2 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 3^{x^2} &= 9^{2x-2} \\ 3^{x^2} &= (3^2)^{2x-2} \\ 3^{x^2} &= 3^{4x-4} \\ x^2 &= 4x - 4 \\ x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

The equation simplifies to $a^X = b^Y$

- Apply your favorite type of log on both sides.
- Bring down the exponents using the third Law of Logarithms
- Finish solving.
- For example:

$$\begin{aligned} 4^{x+1} &= 7^{1-2x} \\ \ln 4^{x+1} &= \ln 7^{1-2x} \\ (x+1)\ln 4 &= (1-2x)\ln 7 \\ x\ln 4 + \ln 4 &= \ln 7 - 2x\ln 7 \\ x\ln 4 + 2x\ln 7 &= \ln 7 - \ln 4 \\ x(\ln 4 + 2\ln 7) &= \ln 7 - \ln 4 \\ x &= \frac{\ln 7 - \ln 4}{\ln 4 + 2\ln 7} \end{aligned}$$

Quadratic-Type Exponential Equations

- Use a substitution to make the equation a quadratic.
- Solve for the new variable.
- Go back to the original variable and finish solving.
- For example:

$$\begin{aligned} e^{2x} - 4e^x - 5 &= 0 \\ (e^x)^2 - 4e^x - 5 &= 0 \\ \text{Make the substitution } y &= e^x \\ y^2 - 4y - 5 &= 0 \\ (y-5)(y+1) &= 0 \\ y = 5 \text{ or } y &= -1 \\ e^x = 5 \text{ or } e^x &= -1 \\ x = \ln 5 \text{ or } x &= \cancel{\ln(-1)} \end{aligned}$$

Examples

1. $4^{3x-5} = 16$

   $\boxed{x = \frac{7}{3}}$

2. $27 = 3^{5x} \cdot 9^{x^2}$

   $\boxed{x = \frac{1}{2} \text{ or } x = -3}$

3. $e^{4t} = 1000$

   $\boxed{t = \frac{\ln 1000}{4}}$

4. $5^{x+2} = 4^{1-x}$

   $\boxed{x = \frac{\ln 4 - \ln 5}{\ln 5 + \ln 4} = \frac{\ln \frac{4}{5}}{\ln 20}}$

5. $e^x + e^{-x} = 4$

   $\boxed{x = \ln(2-\sqrt{3}) \approx -1.317 \text{ or } x = \ln(2+\sqrt{3}) \approx 1.317}$

2 Logarithmic Equations

Definition

- A *logarithmic equation* is one where the variable appears inside a logarithm.
- For all these questions, you must check your answers against the domain of the original equation. There can be fake solutions!
- There are a few possible types that we will be solving.

A logarithm is Raised to a Power

- Solve using a substitution.
- For example:

$$
\begin{aligned}
(\log_3 x)^2 - \log_3 x^2 &= 3 \\
(\log_3 x)^2 - 2\log_3 x &= 3 \\
\text{Make the substitution } y &= \log_3 x \\
y^2 - 2y &= 3 \\
y^2 - 2y - 3 &= 0 \\
(y-3)(y+1) &= 0 \\
y = 3 \text{ or } y &= -1 \\
\log_3 x = 3 \text{ or } \log_3 x &= -1 \\
x = 3^3 \text{ or } x &= 3^{-1} \\
x = 27 \text{ or } x &= \frac{1}{3}
\end{aligned}
$$

Domain restrictions:
$x > 0$ and $x^2 > 0$

- Check $x = 27$:

  $27 > 0$ ✓

  $27^2 > 0$ ✓

- Check $x = \frac{1}{3}$:

  $\frac{1}{3} > 0$ ✓

  $\left(\frac{1}{3}\right)^2 > 0$ ✓

No Logarithms Raised to a Power

1. If there are any constant terms, move all those to one side and the log terms to the other side.
2. Combine all the log terms on each side into a single logarithm using the Laws of Logarithms.
3. At this point, it should be in one of two forms:
   - $\log_a X = Y$. Rewrite as $a^Y = X$ and finish solving.
   - $\log_a X = \log_a Y$. Set $X = Y$ and finish solving.

- For example:

$$
\begin{aligned}
\ln(x+8) + \ln(x-1) &= 2\ln x \\
\ln((x+8)(x-1)) &= \ln x^2 \\
\ln(x^2 + 7x - 8) &= \ln x^2 \\
x^2 + 7x - 8 &= x^2 \\
7x - 8 &= 0 \\
7x &= 8 \\
x &= \frac{8}{7}
\end{aligned}
$$

Domain restrictions:
$x + 8 > 0$, $x - 1 > 0$, $x > 0$

- Check $x = \frac{8}{7}$:

  $\frac{8}{7} + 8 = \frac{64}{7} > 0$ ✓

  $\frac{8}{7} - 1 = \frac{1}{7} > 0$ ✓

  $\frac{8}{7} > 0$ ✓

Examples

1. $\log(4 - x) = 2$

   $\boxed{x = -96}$

2. $\log_3(1 - x) = 1 - \log_3(-x - 1)$

$\boxed{x = -2}$

3. $\ln x + \ln(x-1) = \ln 20$

$\boxed{x = 5}$

4. $\log_3 x^4 = (\log_3 x)^2$

$\boxed{x = 1 \text{ or } x = 81}$