Section 5.6 Notes

Applications and Models: Growth and Decay; Compound Interest

Exponential Growth
A quantity that experiences exponential growth will increase according to the equation

$$\boxed{P(t) = P_0 e^{kt}}$$

where

- t is the time (in any given units)
- $P(t)$ is the amount at time t
- P_0 is the initial quantity.
- k (which needs to be positive) is the exponential growth rate.

A quantity that experiences exponential growth also has a corresponding doubling time. If the doubling time is T, then the population will increase according to the equation

$$P(t) = P_0 e^{kt}, \text{ where } k = \frac{\ln 2}{T}$$

Notice you can also solve for T to get the equation $T = \frac{\ln 2}{k}$

Examples

1. The exponential growth rate of a population of rabbits is 11.6% per month. What is the doubling time?

   About 6 months.

2. A sample of bacteria is growing in a Petri dish. There were originally 2 thousand cells, and after 2 hours there are now 5 thousand cells. How long will it take for there to be 8 thousand cells?

   About 3 hours.

Compounded Interest
An an investment earning continuously compounded interest grows according to the formula:

$$\boxed{P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}}$$

where

- t is the time (in years)
- $P(t)$ is the total amount of money at time t
- P_0 is the principal - or initial amount of the investment.
- r is the interest rate.
- n is the number of times the interest is compounded per year.

  Page 327 has a chart with key words to help figure out what n is.

Continuously Compounded Interest
An an investment earning continuously compounded interest grows according to the formula:

$$\boxed{P(t) = P_0 e^{kt}}$$

where

- t is the time (in years)
- $P(t)$ is the total amount of money at time t
- P_0 is the principal - or initial amount of the investment.
- k is the nominal interest rate.

Notice that this is *exactly* the same as the formula for exponential growth. Problems involving exponential growth and continuously compounded interest work exactly the same.

Example

Suppose that $\$82,000$ is invested at $4\frac{1}{2}\%$ interest, compounded quarterly. Find the function for the amount to which the investmnt grows after t years.

$\boxed{P(t) = 82000(1.01125)^{4t}}$

A father wishes to invest money to help pay for his son's college education. The investment earns 5% compounded continuously. How much should he invest when his son is born so that he'll have $50,000 when his son turns 18?

$\boxed{\$20328.48}$

Exponential Decay
A quantity that experiences exponential decay will decrease according to the equation

$$\boxed{P(t) = P_0 e^{-kt}}$$

where

- t is the time (in any given units)
- $P(t)$ is the amount at time t
- P_0 is the initial quantity.
- k (which needs to be positive) is the decay rate.

A quantity that experiences exponential decay also has a corresponding half-life. If the half-life if T, then the sample will decrease according to the equation

$$P(t) = P_0 e^{-kt}, \text{ where } k = \frac{\ln 2}{T}$$

Notice you can also solve for T to get the equation $T = \frac{\ln 2}{k}$

Example

The half-life of radium-226 is 1600 years. Find the decay rate.

$\boxed{k = \frac{\ln 2}{1600} \approx 0.0004332 = 0.04332\% \text{ per year}}$

Newton's Law of Cooling
An object that's hotter/colder than it's surrounding environment will cool off/heat up according to the equation

$$\boxed{T(t) = T_0 + (T_1 - T_0)e^{-kt}}$$

where

- t is the time (in any given units)
- $T(t)$ is temperature of the object at time t
- T_0 is temperature of the surrounding environment
- T_1 is the initial temperature of the object
- k is a constant that depends on the physical properties of the object and its surrounding. It will change based on how easily they transfer heat to each other.

Example

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. If the temperature is 150° in half an hour, what is the temperature after 45 minutes?

$\boxed{137°\text{F}}$