Section 8.1 Notes

Sequences and Series

1 Sequences

Definition

- A *sequence* is a list of numbers.
- A sequence could be finite, such as:
$$1, 2, 3, 4$$
- A sequence can also be infinite, such as:
$$2, 4, 8, 16, 32, 64, \ldots$$
Here the "$\ldots$" just means that the pattern continues forever.

Sequence Notation

- A series can also be thought of as a function where you only plug in natural numbers.
- For example, "$2, 4, 8, 16, 32, 64, \ldots$" can be thought of as
$$a(n) = 2^n$$
since $a(1) = 2^1 = 2$, $a(2) = 2^2 = 4$, $a(3) = 2^3 = 8$, etc
- Normally, however, the sequence is written with the input variable as a subscript:
$$a_n = 2^n$$
Here $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, etc

Examples

1. Find the first three terms of the sequence $a_n = \frac{n+1}{n}$.

   $\boxed{a_1 = 2,\ a_2 = \frac{3}{2},\ a_3 = \frac{4}{3}}$

2. Find the first four terms of the sequence $a_n = (-1)^n n^3$

   $\boxed{a_1 = -1,\ a_2 = 8,\ a_3 = -27,\ a_4 = 64}$

3. Find the formula for the general term a_n of the sequence $2, 4, 6, 8, 10, \ldots$

   $\boxed{a_n = 2n}$

4. Find the formula for the general term a_n of the sequence $-2, 6, -18, 54, -162, \ldots$

   $\boxed{a_n = (-1)^n \cdot 2 \cdot 3^{n-1}}$

Recursive Sequences

- Some sequences can be written *recursively*, i.e. the terms are calculated from previous terms.
- The most famous example of this type of sequence is the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

- For this sequence, the first two numbers are 0 and 1. Every number after that is calculated by adding the two previous numbers:
  - $a_3 = 0 + 1 = 1$
  - $a_4 = 1 + 1 = 2$
  - $a_5 = 1 + 2 = 3$
  - $a_6 = 2 + 3 = 5$
  - $a_7 = 3 + 5 = 8$
  - $a_8 = 5 + 8 = 13$
  - $a_9 = 8 + 13 = 21$

Examples
Find the first five terms of each sequence.

1. $a_{n+1} = \sqrt{a_n}$, $a_1 = 256$

   $$\boxed{a_1 = 256, a_2 = 16, a_3 = 4, a_4 = 2, a_5 = \sqrt{2}}$$

2. $a_{n+1} = a_n - a_{n-1}$, $a_1 = -10$, $a_2 = 8$

   $$\boxed{a_1 = -10, a_2 = 8, a_3 = 18, a_4 = 10, a_5 = -8}$$

2 Series and Sums

Definition
Suppose you have an infinite sequence a_n. The *nth partial sum* is the sum of the first n terms:

$$\boxed{S_n = a_1 + a_2 + ... + a_n}$$

For example, if $a_n = n + 1$, ($a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$, $a_5 = 6$)

$$S_5 = 2 + 3 + 4 + 5 + 6 = 20$$

Sigma Notation
To represent sums, we use the uppercase Greek letter sigma, $\sum$.
For example:

$$\begin{aligned}\sum_{k=1}^{4}(2k-1) &= (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) \\ &= 1 + 3 + 5 + 7 \\ &= 16\end{aligned}$$

Examples

1. Find the 6th partial sum of the sequence $5, 10, 15, 20, \ldots$

   $\boxed{105}$

2. Evaluate the sum: $\displaystyle\sum_{j=1}^{4} \frac{3}{j+1}$

   $\boxed{\frac{77}{20}}$

3. Write the sum in sigma notation: $-\frac{3}{2} + \frac{4}{4} - \frac{5}{8} + \frac{6}{16} - \frac{7}{32}$

   $$\boxed{\sum_{k=1}^{5} \frac{(-1)^k (k+2)}{2^k}}$$