Section 8.7 Notes

The Binomial Theorem

1 Formulas

Factorial

$$n! = \begin{cases} n(n-1)\dots(2)(1) & \text{if } n = 2, 3, 4, \dots \\ 1 & \text{if } n = 0 \text{ or } 1 \end{cases}$$

For example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Examples

1. Compute $6!$.

   $\boxed{720}$

2. Compute $\dfrac{10!}{8!}$.

   $\boxed{90}$

Binomial Coefficient

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The binomial coefficient is read as "n choose k". The number on the bottom is *never* bigger than the number on top.
For example:

$$\begin{aligned}\binom{5}{3} &= \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} \\ &= \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})(2 \cdot 1)} \\ &= \frac{20}{2} = 10\end{aligned}$$

Example

Compute $\binom{10}{1} + \binom{10}{9}$.

$\boxed{20}$

2 Pascal's Triangle

Construction of Pascal's Triangle
To generate Pascal's Triangle, you start with a 1 at the top. Then, the numbers for each subsequent row is found by adding the two numbers above it. You can imagine that everything "outside" of the triangle is a zero.

$$
\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&20&&15&&6&&1
\end{array}
$$

Relationship to Binomial Coefficients
It turns out, that you can use Pascal's Triangle to compute the binomial coefficients rather than using the formula.

$$
\begin{array}{ccccccccccccc}
&&&&&&\binom{0}{0}=1&&&&&& \\
&&&&&\binom{1}{0}=1&&\binom{1}{1}=1&&&&& \\
&&&&\binom{2}{0}=1&&\binom{2}{1}=2&&\binom{2}{2}=1&&&& \\
&&&\binom{3}{0}=1&&\binom{3}{1}=3&&\binom{3}{2}=3&&\binom{3}{3}=1&&& \\
&&\binom{4}{0}=1&&\binom{4}{1}=4&&\binom{4}{2}=6&&\binom{4}{3}=4&&\binom{4}{4}=1&& \\
&\binom{5}{0}=1&&\binom{5}{1}=5&&\binom{5}{2}=10&&\binom{5}{3}=10&&\binom{5}{4}=5&&\binom{5}{5}=1& \\
\binom{6}{0}=1&&\binom{6}{1}=6&&\binom{6}{2}=15&&\binom{6}{3}=20&&\binom{6}{4}=15&&\binom{6}{5}=6&&\binom{6}{6}=1
\end{array}
$$

Each row of the triangle gives you all the coefficients for a particular n in $\binom{n}{k}$:

$$
\begin{array}{lccccccccccccc}
n=0: &&&&&&&1&&&&&& \\
n=1: &&&&&&1&&1&&&&& \\
n=2: &&&&&1&&2&&1&&&& \\
n=3: &&&&1&&3&&3&&1&&& \\
n=4: &&&1&&4&&6&&4&&1&& \\
n=5: &&1&&5&&10&&10&&5&&1& \\
n=6: &1&&6&&15&&20&&15&&6&&1
\end{array}
$$

3 Multiplying Binomial Expressions

The Binomial Theorem

The Binomial Theorem gives a formula for multiplying out binomial expressions:

$$\begin{aligned}(A+B)^n = &\binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \dots \\ &+ \binom{n}{n-2}A^2B^{n-2} + \binom{n}{n-1}AB^{n-1} + \binom{n}{n}B^n\end{aligned}$$

Examples

1. Expand $(x+y)^3$, and compute the binomial coefficients using the formula.

   $x^3 + 3x^2y + 3xy^2 + y^3$

2. Expand $(2r+s^2)^4$, using Pascal's Triangle to find the coefficients.

   $16r^4 + 32r^3s^2 + 24r^2s^4 + 8rs^6 + s^8$

3. Expand $\left(2a - \frac{1}{2}b\right)^6$

   $64a^6 - 96a^5b + 60a^4b^2 - 20a^3b^3 + \frac{15}{4}a^2b^4 - \frac{3}{8}ab^5 + \frac{1}{64}b^6$

Finding a Single Term
The $(k+1)$st term in $(A+B)^n$ is

$$\binom{n}{k}A^{n-k}B^k$$

Examples

1. Find the fourth term in $(5x-2y)^5$.

   $-2000x^2y^3$

2. Find the tenth term in the expansion of $\left(2x + \frac{y}{2}\right)^{14}$

   $\frac{1001}{8}x^5y^9$