Just-In-Time 10-13 Notes

Sections 10-13

1 JIT 10: Introduction to Polynomials

Algebraic Expressions

- An *algebraic expression* is created by combining numbers and variables using mathematical operations such as addition, subtraction, multiplication, division, exponents, roots, etc

  For example, $2xy - 7$, $\frac{4t^2+1}{t-2}$, $\sqrt{\frac{1}{2}z + 11}$

Polynomials

- A *polynomial* is a special type of algebraic expression.
- A polynomial is built by adding or subtracting *terms*. Each term is either a number or looks like ax^k.
  - a is a number. a is called the *coefficient* of x^k.
  - x is the variable. (Other letters besides x can be used as the variable.)
  - k needs to be a whole number - negative exponents aren't allowed.
- Examples of polynomials:
  - $5x^3 + 1$
  - $-y^2 + \frac{11}{3}y^5 - 3y$
- The *leading term* is the term with the highest exponent.
- The *degree* of the polynomial is the exponent of the leading term.
- The *leading coefficient* is the coefficient of the leading term.
- The *constant term* is the term that is simply a number without a variable.
- The number of terms in a polynomial can be used to categorize it:
  - A polynomial with one term is a *monomial*.
  - A polynomial with two terms is a *binomial*.
  - A polynomial with three terms is a *trinomial*.

Examples
For each polynomial, find the degree, the leading coefficient, the constant term and categorize it by the number of terms.

1. $4x^2 - 3x + 1$

   - Degree: 2
   - Leading Coefficient: 4
   - Constant Term: 1
   - Type of polynomial: trinomial

2. $2x - x^3$

   - Degree: 3
   - Leading Coefficient: -1
   - Constant Term: 0
   - Type of polynomial: binomial

For each polynomial, find the degree, the leading coefficient, the constant term and categorize it by the number of terms.

3. 5

   - Degree: 0
   - Leading Coefficient: 5
   - Constant Term: 5
   - Type of polynomial: monomial

4. $8x^6 - 2x^2 + x - 3x^3$

   - Degree: 6
   - Leading Coefficient: 8
   - Constant Term: 0
   - Type of polynomial: four terms

2 JIT 11: Add and Subtract Polynomials

Like Terms

- *Like terms* of a polynomial are terms where the variables have matching exponents.

  $$\boxed{5x^3} - \boxed{2x^2} + \boxed{3x^3} + x - \boxed{7x^2} + 1$$

  Here, $5x^3$ and $3x^3$ are like terms, as well as $-2x^2$ and $-7x^2$.

- To add or subtract polynomials we "combine like terms" by adding or subtracting the coefficients of the like terms.

Examples
Add or subtract the following polynomials:

1. $(-2x^3 - x + 3) + (7x^2 + 5x - 2)$

   $-2x^3 + 7x^2 + 4x + 1$

2. $(4x^5 - x^3 + x - 1) - (20x^5 + x^4 - 3x^3 + 17x + 5)$

   $-16x^5 - x^4 + 2x^3 - 16x - 6$

3. $(8xy^2 - 3x^3y + xy) + (4xy - 2xy^2 - 5x^2y)$

   $-3x^3y - 5x^2y + 6xy^2 + 5xy$

3 JIT 12: Multiply Polynomials

Multiplying Monomials

To multiply two monomials, multiply their coefficients and add their exponents.

For example:

1. $(5x^3)(2x^6) = 10x^9$
2. $(3x)(-6x^3) = -18x^4$
3. $(4x^2)(-6y) = -24x^2y$ (or $-24yx^2$)

Larger Polynomials

To multiply two binomials, we can FOIL (first, outer, inner, last):

$$(a+b)(c+d) = \overset{\text{F}}{\boxed{ac}} + \overset{\text{O}}{\boxed{ad}} + \overset{\text{I}}{\boxed{bc}} + \overset{\text{L}}{\boxed{bd}}$$

In general, we can multiply expressions with any number of terms by making sure to pair up every term from the first set of parenthesis with every term from the second set.
For example:

$$(a+b+c)(d+e+f) = \underbrace{ad+ae+af}_{\text{pairs with } a} + \underbrace{bd+be+bf}_{\text{pairs with } b} + \underbrace{cd+ce+cf}_{\text{pairs with } c}$$

Examples

Multiply the following expressions and simplify:

1. $(4x-2y)(3y-7x)$

   $\boxed{-28x^2 + 26xy - 6y^2}$

2. $(2s+1)(-s+5)$

   $\boxed{-2s^2 + 9s + 5}$

3. $(x^2+1)^3$

   $\boxed{x^6 + 3x^4 + 3x^2 + 1}$

4 JIT 13: Factor Polynomials

Definition of Factoring

Factoring is the opposite of multiplying out algebraic expressions.
For example, we just figured out that

$$(2s+1)(-s+5) = -2s^2 + 9s + 5$$

by multiplying it out.
This means that $-2s^2 + 9s + 5$ factors into $(2s+1)(-s+5)$.

Greatest Common Factor

- Recall the distributive property: $a(b+c) = ab + ac$. We can use this in reverse to factor.
- For example we can factor $4x^3 + 32x$ by using $a = 4x$, $b = x^2$ and $c = 8$:

$$\begin{aligned} 4x^3 + 32x &= (4x)(x^2) + (4x)(8) \\ &= 4x(x^2 + 8) \end{aligned}$$

  In this example, $4x$ is the *greatest common factor* of $4x^3$ and $32x$.

- Finding the greatest common factor of all the terms:
  1. Find the greatest common factor of the coefficients.
  2. For each variable, use the smallest exponent from among all the terms.

Examples
Factor out the greatest common factor of each expression:

1. $7y^2 - 42y$

   $\boxed{7y(y-6)}$

2. $9ab^2 - 36ab + 18a^3b^2$

   $\boxed{9ab(b-4+2a^2b)}$

3. $51(2x+3)^2 + 17(2x+3)$

   $\boxed{34(2x+3)(3x+5)}$

Factoring by Grouping
This method can sometimes work with expressions having four or more terms.

1. Rearrange the terms into 2 or more groups, each with an equal number of terms.
2. Pull out the greatest common factor for each group.
3. If the "leftovers" are the same for *all* groups, factor out this common term.

Examples
Factor the following expressions by grouping:

1. $x^3 + x^2 - 8x - 8$

   $\boxed{(x+1)(x^2-8)}$

2. $6s^2 + 3st + 8st^2 + 4t^3$

   $\boxed{(2s+t)(3s+4t^2)}$

Factoring Quadratic Trinomials

A *quadratic trinomial* has the form $ax^2 + bx + c$. The methods for factoring quadratic trinomials also work for expressions with the form $ax^2 + bxy + cy^2$.

Method 1: Guess and Check

1. $ax^2 + bx + c$ will factor into the form $\left(\boxed{p}x + \boxed{q}\right)\left(\boxed{r}x + \boxed{s}\right)$ and $ax^2 + bxy + cy^2$ will factor into the form $\left(\boxed{p}x + \boxed{q}y\right)\left(\boxed{r}x + \boxed{s}y\right)$.
2. p and r have to multiply to a and q and s have to multiply to c.
3. Make guesses at p, q, r and s, and FOIL them out to check if they match the original expression.
4. **Special Case:** If the quadratic has the form x^2+bx+c, then it factors into $\left(x + \boxed{q}\right)\left(x + \boxed{s}\right)$ where q and s multiply to c and add to b.

Examples

Factor using the guess and check method.

1. $6x^2 - 2x - 4$.

   $\boxed{2(3x + 2)(x - 1)}$

2. $x^2 + 3x - 10$

   $\boxed{(x + 5)(x - 2)}$

To factor $ax^2 + bx + c$ (or $ax^2 + bxy + cy^2$)

Method 2: ac Method

1. Multiply $a \cdot c$.
2. Find two numbers s and t so that $s \cdot t = a \cdot c$ and $s + t = b$.
3. Rewrite the polynomial as $ax^2 + sx + tx + c$.
4. Factor by grouping.

Examples

Factor using the ac method.

1. $5x^2 - 7x - 6$

   $\boxed{(5x + 3)(x - 2)}$

2. $18x^2 - 3xy - 3y^2$

   $\boxed{3(2x - y)(3x + y)}$

Special Factoring Formulas

1. Difference of Squares: $A^2 - B^2 = (A - B)(A + B)$
2. Difference of Cubes: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
3. Sum of Cubes: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
4. Perfect Squares:
   - $A^2 + 2AB + B^2 = (A + B)^2$
   - $A^2 - 2AB + B^2 = (A - B)^2$

Examples
Factor each by using a special factoring formula.

1. $54x^3 + 2y^6$

   $\boxed{2(3x + y^2)(9x^2 - 3xy^2 + y^4)}$

2. $(2x + 1)^2 - 81$

   $\boxed{4(x - 4)(x + 5)}$

3. $5x^2y^4 - 5x^2z^4$

   $\boxed{5x^2(y - z)(y + z)(y^2 + z^2)}$