Section 1.1 Notes

Real Numbers

1 Types of Real Numbers

The Natural Numbers

- $1, 2, 3, 4, 5, 6, \ldots$
- These are also sometimes called "counting numbers".
- Denoted by the symbol $\mathbb{N}$

Integers

- $\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots$
- Denoted by the symbol $\mathbb{Z}$ (This comes from the German word zählen, which means "to count").

Rationals

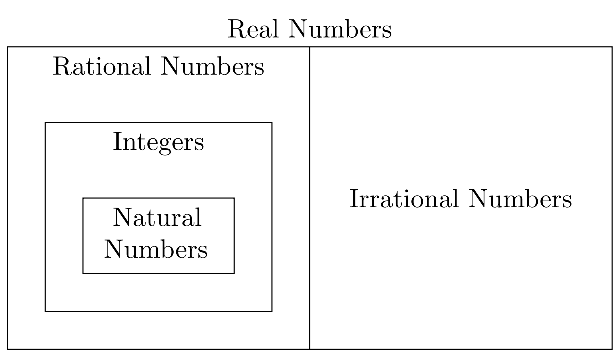
- A rational number is a fraction of two *integers*. For example, $\frac{2}{5}$.
- Denoted by the symbol $\mathbb{Q}$ (This comes from the word "quotient")
- Every rational number has a decimal expansion that either *terminates* (stops) or *repeats*.
- More examples of rational numbers:

1. $-\frac{17}{8} = \frac{-17}{8}$
2. $2 = \frac{2}{1}$
3. $2.47 = \frac{247}{100}$
4. $0.33333\cdots = \frac{1}{3}$

Irrationals

- An irrational number is any other real number that isn't a rational number.
- The decimal expansion of irrational numbers is *infinite* and *nonrepeating*.
- Examples of irrational numbers: $\pi \approx 3.14159265\ldots$ and $\sqrt{2} \approx 1.41421356\ldots$

Real Numbers



Example
Which of the following numbers are rational?

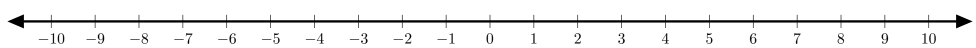
$\frac{\pi}{4}, -\sqrt{16}, 15.75, \frac{1}{\sqrt{17}}, 12$

Answer:

$$-\sqrt{16} = \frac{-4}{1}$$
$$15.75 = \frac{1575}{100}$$
$$12 = \frac{12}{1}$$

2 Sets of Real Numbers

Real Number Line
The real number line is a visual representation of all the real numbers along a line, arranged into their normal order.



Intervals

An interval is a section of the real number line.

Example:

Finite Intervals
A finite interval is a section of the number line between two endpoints:

The set of all real numbers between 2 and 3, excluding both 2 and 3.

Algebraic Notatation: $2 < x < 3$
Set Notation: $\{x \mid 2 < x < 3\}$
Interval Notation: $(2, 3)$

The set of all real numbers between -17 and 4, including both -17 and 4.

Algebraic Notatation: $-17 \leq x \leq 4$
Set Notation: $\{x \mid -17 \leq x \leq 4\}$
Interval Notation: $[-17, 4]$

The set of all real numbers between $\frac{3}{8}$ and 12, including $\frac{3}{8}$ and excluding 12.

Algebraic Notatation: $\frac{3}{8} \leq x < 12$
Set Notation: $\left\{x \mid \frac{3}{8} \leq x < 12\right\}$
Interval Notation: $\left[\frac{3}{8}, 12\right)$

The set of all real numbers between -6 and π, excluding -6 and including π.

Algebraic Notatation: $-6 < x \leq \pi$
Set Notation: $\{x \mid -6 < x \leq \pi\}$
Interval Notation: $(-6, \pi]$

Infinite Intervals

An infinite interval is a section of the real number line before or after a single endpoint.

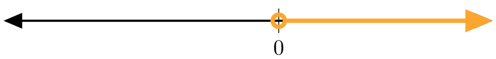
The set of all real numbers smaller than 5, excluding 5.

Algebraic Notatation: $x < 5$
Set Notation: $\{x \mid x < 5\}$
Interval Notation: $(-\infty, 5)$

The set of all real numbers smaller than -100, including -100.

Algebraic Notatation: $x \leq -100$
Set Notation: $\{x \mid x \leq -100\}$
Interval Notation: $(-\infty, -100]$

The set of all real numbers bigger than 0, excluding 0.



Algebraic Notatation: $x > 0$
Set Notation: $\{x \mid x > 0\}$
Interval Notation: $(0, \infty)$

The set of all real numbers bigger than 2, including 2.

Algebraic Notatation: $x \geq 2$
Set Notation: $\{x \mid x \geq 2\}$
Interval Notation: $[2, \infty)$

Special Sets

- The set of all real numbers.

  Interval Notation: $(-\infty, \infty)$

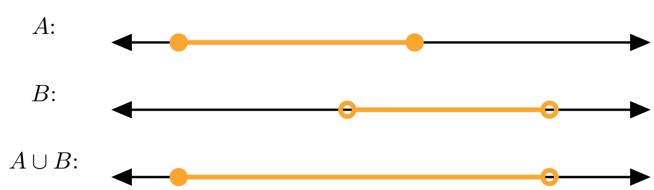
- The empty set.

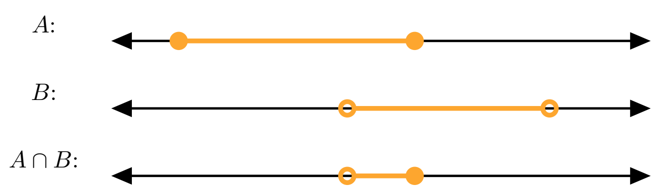
  This is denoted with the special symbol $\varnothing$.

Set Operations

- Union: If A and B are two sets, the union of A and B (denoted $A \cup B$) is the set containing any number that's in either A or B (or both).

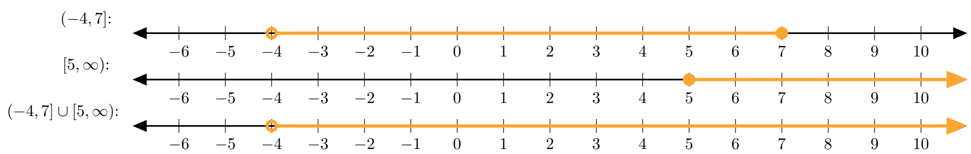
- Intersection: If A and B are two sets, the intersection of A and B (denoted $A \cap B$) is the set containing only those numbers in both A and B.

Example

Simplify $(-4, 7] \cup [5, \infty)$.

Answer:

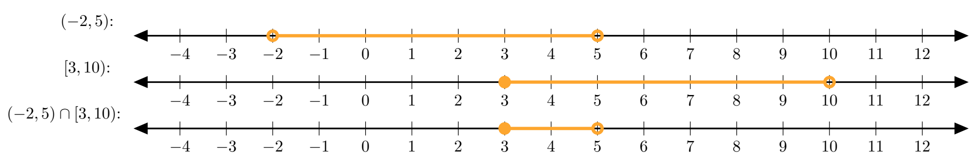


$$\boxed{(-4, \infty)}$$

Example

Simplify $(-2, 5) \cap [3, 10)$.

Answer:



$$\boxed{[3, 5)}$$

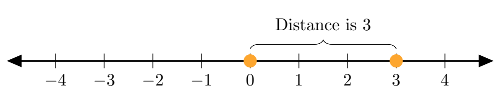
3 Absolute Value

Definition

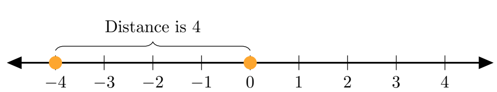
The absolute value is defined algebraically as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Geometrically, it gives the distance between x and 0 on the number line:



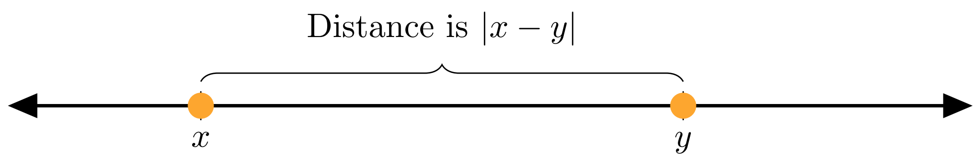
Therefore, $|3| = 3$.



Therefore, $|-4| = 4$.

Distance on the Number Line
If x and y are two numbers on the number line, the distance between them is

$$|x-y|$$



Examples

1. Find the distance between 3 and 6.

   Answer: $\boxed{3}$

2. Find the distance between $\frac{12}{5}$ and $\frac{9}{4}$.

   Answer: $\boxed{\frac{3}{20}}$

3. Write the following statement as an inequality: "x is at least 3 units from y."

   Answer: $\boxed{|x-y| \geq 3}$

4 Fractions

Adding or Subtracting Fractions

1. Get a *common denominator* by multiplying both the top and bottom of each fraction by an appropriate number.
2. Add or subtract the numerators, keeping the common denominator.
3. Simplify the fraction by canceling out common factors from the top and bottom.

Example: $\frac{1}{12} - \frac{5}{28} = -\frac{2}{21}$
Work:

$$12 = 4 \cdot 3$$
$$28 = 4 \cdot 7$$
$$\frac{1}{12} \cdot \frac{7}{7} - \frac{5}{28} \cdot \frac{3}{3} = \frac{7}{84} - \frac{15}{84} = \frac{-8}{84} = -\frac{2}{21}$$

Multiplying Fractions

1. Cancel out common factors. You may cancel factors in either numerator with factors in either denominator.
2. Multiply across the numerators and across the denominators.

Example: $\frac{2}{7} \cdot \frac{21}{8} = \frac{3}{4}$
Work:

$$\frac{2}{7} \cdot \frac{21}{8} = \frac{1}{1} \cdot \frac{3}{4} = \frac{3}{4}$$

Dividing Fractions

1. Change it to a multiplication problem:

$$\tfrac{a}{b} \div \tfrac{c}{d} = \tfrac{a}{b} \cdot \tfrac{d}{c} \quad \text{or} \quad \tfrac{\frac{a}{b}}{\frac{c}{d}} = \tfrac{a}{b} \cdot \tfrac{d}{c}$$

2. Finish it just like a multiplication problem.

Example: $\frac{42}{5} \div \frac{35}{2} = \frac{12}{25}$
Work:

$$\frac{42}{5} \div \frac{35}{2} = \frac{42}{5} \cdot \frac{2}{35} = \frac{6}{5} \cdot \frac{2}{5} = \frac{12}{25}$$

Examples
Simplify the following:

1. $\dfrac{-\frac{1}{15}}{\frac{2}{5} - \frac{4}{9}}$

   Answer: $\boxed{\dfrac{3}{2}}$

2. $\dfrac{2 - \frac{10}{11}}{\frac{7}{22} - \frac{3}{4}}$

   Answer: $\boxed{-\frac{48}{19}}$

5 Repeating Decimals

Notation
Remember that some rational numbers (fractions of integers) have decimal expansions with an infinite, repeating pattern. For example, $\frac{1}{3} = 0.33333\ldots$.
We will write these with *bar notation*:

- $341.213333\cdots = 341.21\overline{3}$
- $0.878787\cdots = 0.\overline{87}$
- $48.1526526526\cdots = 48.1\overline{526}$

Converting Repeating Decimals to Fractions
Steps:

1. Set $x =$ number.

$$x = 1.2333\ldots$$

2. Multiply both sides of the equation from step 1 by an appropriate power of 10 to place the decimal right before the repetition.

$$10(x) = 10(1.2333\ldots)$$

$$10x = 12.333\ldots$$

3. Multiply both sides of the equation from step 1 by an appropriate power of 10 to place the decimal after a full cycle of the repetition.

$$100(x) = 100(1.2333\ldots)$$

$$100x = 123.333\ldots$$

4. Subtract the equations from steps 2 and 3. Solve for x.

$$\begin{array}{r} 100x = 123.333 \\ 10x = 12.333 \\ \hline 90x = 111 \end{array}$$

$$x = \frac{111}{90} = \frac{37}{30}$$

Examples
Convert the repeating decimal to a fraction of two integers in lowest terms.

1. $0.\overline{42}$

   Answer: $\boxed{\dfrac{14}{33}}$

2. $2.344\overline{1}$

   Answer: $\boxed{\dfrac{21097}{9000}}$

6 Properties of Real Numbers

Commutative Properties

Commutative Property of Addition:

$$x + y = y + x$$

For example, $5 + 3 = 8$ and $3 + 5 = 8$.

Commutative Property of Multiplication:

$$xy = yx$$

For example, $3 \cdot 4 = 12$ and $4 \cdot 3 = 12$.

Associative Properties

Associative Property of Addition:

$$x + (y + z) = (x + y) + z$$

For example, $2 + (3 + (-1)) = 2 + 2 = 4$ and $(2 + 3) + (-1) = 5 - 1 = 4$.

Associative Property of Multiplication:

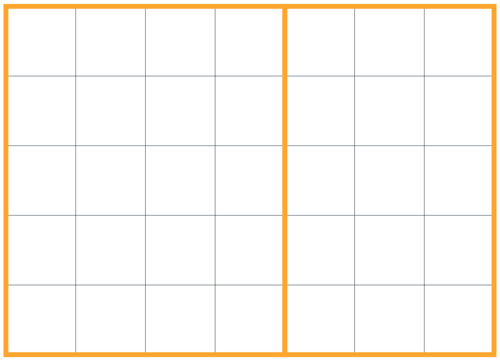
$$x(yz) = (xy)z$$

For example, $(2 \cdot 3) \cdot \frac{1}{2} = 6 \cdot \frac{1}{2} = \frac{6}{2} = 3$ and $2 \cdot \left(3 \cdot \dfrac{1}{2}\right) = 2 \cdot \dfrac{3}{2} = \dfrac{6}{2} = 3$.

Distributive Property

$$x(y+z) = xy + yz$$

For example, $5 \cdot (4+3) = 5 \cdot 7 = 35$ and $5 \cdot 4 + 5 \cdot 3 = 20 + 15 = 35$. The reasoning behind why this property works can be understood by calculating the area of the big rectangle in two ways:



By multiplying the lengths of the big rectangle:

$$5(7) = 5(4+3)$$

By adding the areas of the two smaller rectangles:

$$20 + 15 = 5(4) + 5(3)$$

Examples

Identify which property is being used.

1. $(5-4)+3 = 5+(-4+3)$

   Associative Property of Addition

2. $3x(2+x) = 3x(2) + 3x(x)$

   Distributive Property

3. $(ab)c = c(ab)$

   Commutative Property of Multiplication