Section 1.2 Notes

Exponents and Radicals

1 Exponents

Natural Number Exponents

- If n is a natural number,

$$x^n = \underbrace{x \cdot x \cdot x \ldots x}_{n \text{ times}}$$

- When we use this notation, x is called the *base* and n is called the *exponent*.
- Examples:

1. $9^2 = 9 \cdot 9 = 81$
2. $2^3 = 2 \cdot 2 \cdot 2 = 4 \cdot 2 = 8$
3. $(a+2)^5 = (a+2)(a+2)(a+2)(a+2)(a+2)$

Integer Exponents

- If m is an integer, $x^{-m} = \frac{1}{x^m}$.
- Examples:

1. $2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{2 \cdot 2 \cdot 2 \cdot 2} = \dfrac{1}{16}$
2. $\dfrac{1}{5^{-3}} = 5^{-(-3)} = 5^3 = 125$

Laws of Exponents

1. $x^a \cdot x^b = x^{a+b}$

$$2^4 \cdot 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$$

2. $\frac{x^a}{x^b} = x^{a-b}$

$$\frac{4^5}{4^3} = \frac{4 \cdot 4 \cdot \cancel{4} \cdot \cancel{4} \cdot \cancel{4}}{\cancel{4} \cdot \cancel{4} \cdot \cancel{4}} = 4^2$$

3. $(x^a)^b = x^{ab}$

$$(5^4)^3 = 5^4 \cdot 5^4 \cdot 5^4 = 5^{4+4+4} = 5^{12}$$

4. $(xy)^a = x^a y^a$

$$(2z)^5 = 2z \cdot 2z \cdot 2z \cdot 2z \cdot 2z = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot z \cdot z \cdot z \cdot z \cdot z = 2^5 z^5$$

5. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

$$\left(\frac{7}{2}\right)^3 = \frac{7}{2} \cdot \frac{7}{2} \cdot \frac{7}{2} = \frac{7 \cdot 7 \cdot 7}{2 \cdot 2 \cdot 2} = \frac{7^3}{2^3}$$

6. $\left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^{a}$

$$\left(\frac{4}{3}\right)^{-2} = \frac{4^{-2}}{3^{-2}} = \frac{\frac{1}{4^2}}{\cdot}\,\frac{1}{3^2} = \frac{1}{4^2} \cdot \frac{3^2}{1} = \frac{3^2}{4^2}$$

7. $\frac{x^{-a}}{y^{-b}} = \frac{y^b}{x^a}$

$$\frac{5^{-3}}{2^{-4}} = \frac{\frac{1}{5^3}}{\frac{1}{2^4}} = \frac{1}{5^3} \cdot \frac{2^4}{1} = \frac{2^4}{5^3}$$

Special Cases

- $x^1 = x$
- $x^0 = 1$ as long as $x \neq 0$. (0^0 is undefined.)

Examples

Simplify the following expressions and eliminate any negative exponents.

1. $\dfrac{3x^2y^{-4}}{9x^3y^{-10}}$

   Answer: $\boxed{\dfrac{y^6}{3x}}$

2. $\left(\dfrac{2q^{-4}r^{-3}s}{3r^4s^{-3}}\right)^{-2}$

   Answer: $\boxed{\dfrac{9r^{14}q^8}{4s^8}}$

3. $\left(\dfrac{2a^2}{b}\right)^4 \left(\dfrac{b^2}{4a^3}\right)^2$

   Answer: $\boxed{a^2}$

2 Roots/Radicals

Definition of nth Roots

If n is a natural number and $y^n = x$, then $\sqrt[n]{x} = y$.
$\sqrt[n]{x}$ is read as "the nth root of x".

- $\sqrt[3]{8} = 2$ because $2^3 = 8$.
- $\sqrt[4]{81} = 3$ because $3^4 = 81$.

When $n = 2$ we call it a "square root". However, instead of writing $\sqrt[2]{x}$, we drop the 2 and just write $\sqrt{x}$. So, for example:

- $\sqrt{16} = 4$ because $4^2 = 16$.

Properties of Roots/Radicals

- $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
  (When n is even, x and y need to be nonnegative.)

- $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

  (When n is even, x and y need to be nonnegative.)

- $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$
- $\sqrt[n]{x^n} = x$ if n is odd.
- $\sqrt[n]{x^n} = |x|$ if n is even.

Examples

- $\sqrt[3]{375} - \sqrt[3]{81}$

  Answer: $\boxed{2\sqrt[3]{3}}$

- $\sqrt[5]{x^3y^4}\,\sqrt[5]{x^7y}$

  Answer: $\boxed{x^2y}$

- $\sqrt[3]{\frac{16ab}{a^4b^3}}$

  Answer: $\boxed{\frac{2}{a}\sqrt[3]{\frac{2}{b^2}}}$

3 Rational Exponents

Definition of Rational Exponents
Every nth root has an equivalent exponential form:

$$\sqrt[n]{x} = x^{1/n}$$

When roots are written in their exponential form, you can use all of the Laws of Exponents to simplify problems.
Another nice formula to convert is

$$\sqrt[n]{x^m} = x^{m/n}$$

Examples
Simplify the following expressions, writing your final answer without negative exponents. Assume all variables denote positive quantities.

1. $\sqrt[3]{a^4b^{-1}}\sqrt[9]{a^6b^2}$

   Answer: $\boxed{\dfrac{a^2}{b^{1/9}}}$

2. $\left(\dfrac{r^8s^{-4}}{16s^{4/3}}\right)^{-1/4}$

   Answer: $\boxed{\dfrac{2s^{4/3}}{r^2}}$

3. $\dfrac{\left(27w^6z^{-12}\right)^{-1/6}}{\left(243w^{-5}z^{10}\right)^{1/5}}$

   Answer: $\boxed{\dfrac{1}{3^{3/2}}}$

4 Scientific Notation

Definition of Scientific Notation
A number is in scientific notation if it is written in the form

$$a \times 10^n$$

where a has exactly one non-zero digit left of the decimal point, and n is an integer.
For example: 2.37×10^6 and -1.0021×10^{-100}.

Examples
Convert the following numbers from scientific notation to decimal notation:

1. -3.001×10^5

   Answer: $\boxed{\text{-300,100}}$

2. 2.42×10^{-3}

   Answer: $\boxed{0.00242}$

Examples
Convert the following numbers from decimal notation to scientific notation:

1. $25,040,000$

   Answer: $\boxed{2.504 \times 10^7}$

2. 1.3

   Answer: $\boxed{1.3 \times 10^0}$

3. -0.09624

   Answer: $\boxed{-9.624 \times 10^{-2}}$