

Section 1.3 Notes

Algebraic Expressions

1 Definitions

Algebraic Expressions

- An *algebraic expression* is created by combining numbers and variables using mathematical operations such as addition, subtraction, multiplication, division, exponents, roots, etc

For example, $2xy - 7$, $\frac{4t^2+1}{t-2}$, $\sqrt{\frac{1}{2}z + 11}$

Polynomials

- A *polynomial* is a special type of algebraic expression.
- A polynomial is built by adding or subtracting *terms*. Each term is either a number or looks like ax^k .
 - a is a number. a is called the *coefficient* of x^k .
 - x is the variable. (Other letters besides x can be used as the variable.)
 - k needs to be a whole number - negative exponents aren't allowed.
- Examples of polynomials:
 - $5x^3 + 1$
 - $-y^2 + \frac{11}{3}y^5 - 3y$
- The *leading term* is the term with the highest exponent.
- The *degree* is the exponent of the leading term.
- The *leading coefficient* is the coefficient of the leading term.
- The *constant term* is the term that is simply a number without a variable.
- The number of terms in a polynomial can be used to categorize it:
 - A polynomial with one term is a *monomial*.
 - A polynomial with two terms is a *binomial*.
 - A polynomial with three terms is a *trinomial*.

Examples

For each polynomial, find the degree, the leading coefficient, the constant term and categorize it by the number of terms.

1. $4x^2 - 3x + 1$

- Degree: 2
- Leading Coefficient: 4
- Constant Term: 1
- Type of polynomial: trinomial

2. $2x - x^3$

- Degree: 3
- Leading Coefficient: -1
- Constant Term: 0
- Type of polynomial: binomial

For each polynomial, find the degree, the leading coefficient, the constant term and categorize it by the number of terms.

3. 5

- Degree: 0
- Leading Coefficient: 5
- Constant Term: 5
- Type of polynomial: monomial

4. $8x^6 - 2x^2 + x - 3x^3$

- Degree: 6
- Leading Coefficient: 8
- Constant Term: 0
- Type of polynomial: four terms

2 Adding and Subtracting Polynomials

Like Terms

- *Like terms* of a polynomial are terms where the variables have matching exponents.

$$5x^3 - 2x^2 + 3x^3 + x - 7x^2 + 1$$

Here, $5x^3$ and $3x^3$ are like terms, as well as $-2x^2$ and $-7x^2$.

- To add or subtract polynomials we “combine like terms” by adding or subtracting the coefficients of the like terms.

Examples

Add or subtract the following polynomials:

1. $(-2x^3 - x + 3) + (7x^2 + 5x - 2)$

Answer: $-2x^3 + 7x^2 + 4x + 1$

2. $(4x^5 - x^3 + x - 1) - (20x^5 + x^4 - 3x^3 + 17x + 5)$

Answer: $-16x^5 - x^4 + 2x^3 - 16x - 6$

3 Multiplying Algebraic Expressions

FOIL Method

To multiply two binomials, we can FOIL (first, outer, inner, last):

$$(a + b)(c + d) = \overset{\text{F}}{a}\overset{\text{O}}{c} + \overset{\text{F}}{a}\overset{\text{I}}{d} + \overset{\text{O}}{b}\overset{\text{I}}{d} + \overset{\text{L}}{b}\overset{\text{L}}{d}$$

In general, we can multiply expressions with any number of terms by making sure to pair up every term from the first set of parenthesis with every term from the second set. For example:

$$(a + b + c)(d + e + f) = \underbrace{ad + ae + af}_{\text{pairs with } a} + \underbrace{bd + be + bf}_{\text{pairs with } b} + \underbrace{cd + ce + cf}_{\text{pairs with } c}$$

Examples

Multiply the following expressions and simplify:

1. $(4x - 2y)(3y - 7x)$

Answer: $-28x^2 + 26xy - 6y^2$

2. $(2s + 1)(-s + 5)$

Answer: $-2s^2 + 9s + 5$

3. $(x^2 + 1)^3$

Answer: $x^6 + 3x^4 + 3x^2 + 1$

4 Factoring

Definition of Factoring

Factoring is the opposite of multiplying out algebraic expressions.

For example, we just figured out that

$$(2s + 1)(-s + 5) = -2s^2 + 9s + 5$$

by multiplying it out. This means that $-2s^2 + 9s + 5$ factors into $(2s + 1)(-s + 5)$.

Greatest Common Factor

- Recall the distributive property: $a(b + c) = ab + ac$. We can use this in reverse to factor.
- For example we can factor $4x^3 + 32x$ by using $a = 4x$, $b = x^2$ and $c = 8$:

$$\begin{aligned} 4x^3 + 32x &= (4x)(x^2) + (4x)(8) \\ &= 4x(x^2 + 8) \end{aligned}$$

In this example, $4x$ is the *greatest common factor* of $4x^3$ and $32x$.

- Finding the greatest common factor of all the terms:
 1. Find the greatest common factor of the coefficients.
 2. For each variable, use the smallest exponent from among all the terms.

Examples

Factor out the greatest common factor of each expression:

1. $7y^2 - 42y$

Answer: $7y(y - 6)$

2. $9ab^2 - 36ab + 18a^3b^2$

Answer: $9ab(b - 4 + 2a^2b)$

3. $51(2x + 3)^2 + 17(2x + 3)$

Answer: $34(2x + 3)(3x + 5)$

Factoring by Grouping

This method can sometimes work with expressions having four or more terms.

1. Rearrange the terms into 2 or more groups, each with an equal number of terms.
2. Pull out the greatest common factor for each group.
3. If the “leftovers” are the same for *all* groups, factor out this common term.

Examples

Factor the following expressions by grouping:

1. $x^3 + x^2 - 8x - 8$

Answer: $(x + 1)(x^2 - 8)$

2. $6s^2 + 3st + 8st^2 + 4t^3$

Answer: $(2s + t)(3s + 4t^2)$

Factoring Quadratic Trinomials

A *quadratic trinomial* has the form $ax^2 + bx + c$. The methods for factoring quadratic trinomials also work for expressions with the form $ax^2 + bxy + cy^2$.

Method 1: Guess and Check

1. $ax^2 + bx + c$ will factor into the form $(\boxed{p}x + \boxed{q})(\boxed{r}x + \boxed{s})$ and $ax^2 + bxy + cy^2$ will factor into the form $(\boxed{p}x + \boxed{q}y)(\boxed{r}x + \boxed{s}y)$.
2. p and r have to multiply to a and q and s have to multiply to c .

3. Make guesses at p , q , r and s , and FOIL them out to check if they match the original expression.
4. **Special Case:** If the quadratic has the form x^2+bx+c , then it factors into $(x + \boxed{q})(x + \boxed{s})$ where q and s multiply to c and add to b .

Examples

Factor using the guess and check method.

1. $3x^2 - x - 2$.

Answer: $(3x + 2)(x - 1)$

2. $x^2 + 3x - 10$

Answer: $(x + 5)(x - 2)$

Method 2: ac Method

1. Multiply $a \cdot c$.
2. Find two numbers s and t so that $s \cdot t = a \cdot c$ and $s + t = b$.
3. Rewrite the polynomial as $ax^2 + sx + tx + c$.
4. Factor by grouping.

Examples

Factor using the ac method.

1. $5x^2 - 7x - 6$

Answer: $(5x + 3)(x - 2)$

2. $6x^2 - xy - y^2$

Answer: $(2x - y)(3x + y)$

Special Factoring Formulas

1. Difference of Squares: $A^2 - B^2 = (A - B)(A + B)$
2. Difference of Cubes: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
3. Sum of Cubes: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
4. Perfect Square: $A^2 + 2AB + B^2 = (A + B)^2$

Examples

Factor each by using a special factoring formula.

1. $27x^3 + y^6$

Answer: $(3x + y^2)(9x^2 - 3xy^2 + y^4)$

2. $(2x + 1)^2 - 81$

Answer: $4(x - 4)(x + 5)$

Factoring Using Substitution

Whenever you see an expression being repeated, you can make a *substitution* by replacing the entire expression with a new variable.

You *must* make sure that the variable you pick doesn't show up anywhere else in the problem.

Once you've factored with the new variable, put the original expression back in.

Examples

Factor the following using a substitution.

1. $(a - 2b)^2 - 6(a - 2b) + 5$

Answer: $(a - 2b - 5)(a - 2b - 1)$

2. $5(4x + 3)^2 - 19(4x + 3) - 4$

Answer: $4(5x + 4)(4x - 1)$

Factoring Expressions with Rational Exponents

1. Make any necessary substitutions.
2. Pull out the GCF of the coefficients.
3. For each variable, use the lowest exponent and pull that out. Note: Negative exponents will show up in these problems. The word "lowest" here means "farthest left on the number line". In other words, since $-10 < -1$, we consider -10 to be "lower" than -1 .
4. Simplify and factor whatever is "leftover."

Examples

Factor the following.

1. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$

Answer: $\frac{1}{3}(x^2 + 3)^{-4/3}(x^2 + 9)$

2. $3(x + 1)^{3/2} - 9(x + 1)^{1/2} - 30(x + 1)^{-1/2}$

Answer: $3(x + 1)^{-1/2}(x - 4)(x + 3)$