Section 1.5 Notes

Equations

1 Principles of Equation Solving

Equation Solving Techniques

1. You can add or subtract any number from both sides of an equation.
2. You can multiply or divide by any nonzero number on both sides of an equation.
3. If two numbers multiply to zero, at least one of those numbers must be zero.
4. If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$ (as long as a isn't negative).

2 Linear Equations

Definition

- A *linear equation* is one that can be simplified to the form $ax + b = 0$.
- x is the variable - but other variables are fine too
- a and b are numbers, but $a \neq 0$

Example
Solve the linear equation.

$$9x + 3 = 6(2 - x) + 3x$$

$$\boxed{x = \frac{3}{4}}$$

3 Quadratic Equations

Definition

- A *quadratic equation* is one that can be simplified to the form $ax^2 + bx + c = 0$.
- x is the variable - but other variables are fine too
- a, b and c are numbers, but $a \neq 0$

Quadratic Formula
The equation $ax^2 + bx + c = 0$ can be solved with the following formula:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The portion under the root $(b^2 - 4ac)$ is called the *discriminant*:

- If the discriminant is negative, there are no real solutions (because you cannot take the square root of a negative number).
- If the discriminant is zero, there is exactly on real solution.
- If the discriminant is positive, there are exactly two real solutions.

Example
Solve the quadratic equation.

$$2x^2 - 3x - 5 = 0$$

$$\boxed{x = \tfrac{5}{2},\ x = -1}$$

Factoring
If the equation can be rewritten as $(ax+b)(cx+d) = 0$, then the equations $ax+b = 0$ and $cx+d = 0$ give the solutions to the equation.

- The reason this method works comes from the third equation solving property: "if two numbers multiply to zero, at least one of those numbers must be zero."
- Be aware that not every quadratic can be factored, and so this method will often fail. However, you can *always* use the quadratic formula.

Example
Solve the quadratic equation.

$$10x^2 + x - 3 = 0$$

$$\boxed{x = -\tfrac{3}{5},\ x = \tfrac{1}{2}}$$

Completing the Square
Let's illustrate the steps with an example: $4x^2 + 16x - 9 = 0$

1. Move the constant term to the other side.

$$4x^2 + 16x = 9$$

2. Divide everything by the coefficient of x^2.

$$x^2 + 4x = \frac{9}{4}$$

3. The equation should look like $x^2 + Bx = C$. Do some scratch work to compute:

(a) $\frac{B}{2}$

(b) $\left(\frac{B}{2}\right)^2$

$$(a)\ \frac{4}{2} = 2$$
$$(b)\ 2^2 = 4$$

4. Add $\left(\frac{B}{2}\right)^2$ to both sides of the equation.

$$x^2 + 4x + 4 = \frac{9}{4} + 4$$
$$x^2 + 4x + 4 = \frac{9}{4} + \frac{16}{4}$$
$$x^2 + 4x + 4 = \frac{25}{4}$$

5. The left-hand-side (LHS) of the equation will now *always* factor into $\left(x + \frac{B}{2}\right)^2$.

$$(x+2)^2 = \frac{25}{4}$$

6. Solve using the final equation solving technique: "If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$."

$$x + 2 = \pm\sqrt{\frac{25}{4}}$$
$$x + 2 = \pm\frac{5}{2}$$
$$x = -2 \pm \frac{5}{2} = -\frac{4}{2} \pm \frac{5}{2} = \frac{-4 \pm 5}{2}$$

$x = \frac{-4+5}{2} = \boxed{\frac{1}{2}}$ or $x = \frac{-4-5}{2} = \boxed{-\frac{9}{2}}$.

Example
Solve the quadratic equation.

$$3x^2 + 8x + 1 = 0$$

$$\boxed{x = \frac{-4 \pm \sqrt{13}}{3}}$$

4 Quadratic-Type Equations

Definition
A *quadratic-type* equation has three properties:

1. Three terms
2. Two terms have variables - the third should not
3. The exponent on one of the variables is twice the other.

Solving using Substitution
Let's illustrate with $4x^4 - 13x^2 + 9 = 0$.

1. Figure out which exponent is twice the other. Rewrite as a square.

$$4(\boxed{x^2})^2 - 13\boxed{x^2} + 9 = 0$$

2. Make a substitution for the repeated expression.

$$A = \boxed{x^2}$$
$$4A^2 - 13A + 9 = 0$$

3. Solve for the new variable.

$$(4A - 9)(A - 1) = 0$$
$$4A - 9 = 0 \text{ or } A - 1 = 0$$
$$A = \frac{9}{4} \text{ or } A = 1$$

4. Replace the original variable and solve.

$$x^2 = \frac{9}{4} \text{ or } x^2 = 1$$
$$x = \pm\sqrt{\frac{9}{4}} \text{ or } x = \pm\sqrt{1}$$
$$x = \pm\frac{3}{2} \text{ or } x = \pm 1$$

Example
Solve for y:

$$6y - y^{1/2} - 1 = 0$$

$$\boxed{y = \tfrac{1}{4}}$$

5 Rational Equations

Method

1. Find the least common denominator for every fraction and multiply both sides of the equation by this. This should get rid of all the fractions.

2. Solve the equation to find *potential* solutions.

3. Some of the answers you found might not be solutions! Check your answers against the domain of the original problem. Since the original problem has fractions, you have to make sure you never divide by zero.

Example
Solve for x:

$$\frac{x+3}{x-2}+\frac{3}{x-3}=\frac{-5}{x^2-5x+6}$$

$$\boxed{x=-5}$$

6 Square Root Equations

Method

1. Isolate the root on one side of the equation.
2. Square both sides to cancel out the root.
3. Solve the equation to find *potential* solutions.
4. Check the answers in the original equation - some of them might not actually be solutions!

Example
Solve for x:

$$\sqrt{4x+8}+3x=2x-1$$

$$\boxed{x=-2}$$

7 Solve for One Variable in Terms of Others

Examples

1. Solve for y: $\dfrac{xy+z}{wy+z}=3$

   $\boxed{y=\dfrac{2z}{x-3w}}$

2. Solve for c: $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$

   $\boxed{c=\dfrac{ab}{a+b}}$