Section 1.7 Notes

Inequalities

1 Principles of Inequalities

Definition
An *inequality* is a mathematical relation that compares two expressions using "$<$" (less than), "$>$" (greater than), "$\leq$" (less than or equal to), "$\geq$" (greater than or equal to), or "$\neq$" (not equal to). For practical purposes, in this section we'll focus on "$<$", "$>$", "$\leq$", and "$\geq$" because inequalties with "$\neq$" can be solved almost identically to equations (with $=$).

Inquality Solving Techniques

1. You can add or subtract any number from both sides of an inequality.
2. You can multiply or divide both sides of an iequality by any positive number.
3. You can multiply or divide both sides of an inequality by any negative number, *however*, you must then flip the inequality.
   - "$<$" flips with "$>$"
   - "$\leq$" flips with "$\geq$"

   Don't multiply both sides of an inequality by a variable! Since we don't know if the variable is positive or negative, we don't know whether to flip the inequality or not.

2 Linear Inequalties

Types

1. Simple/"Normal" Linear Inequalities
   - These have two sides to the inequality, e.g. $3x - 1 > 10$.
   - These are solved almost identically to linear equations, except making sure to flip the inequality if you have to divide/multiply by a negative number.
2. Compound Linear Inequalities.
   - These have three "sides" to the inequality, e.g. $-1 < x + 5 \leq 7$.
   - Focus on isolating the variable in the middle.
   - Whatever you do to one "side" you must do to the other two "sides".
   - If you multiply or divide by a negative number, both inequalities flip.

Examples

1. Solve the inequality: $4x < 8x + 7$

   Answer: $\boxed{x > -\frac{7}{4}}$ or $\boxed{(-\frac{7}{4}, \infty)}$

2. Solve the inequality: $2 < -5x - 3 \leq 17$

   Answer: $\boxed{-4 \leq x < 1}$ or $\boxed{[-4, 1)}$

3. The conversion between Fahrenheit and Celsius is given by $F = \frac{9}{5}C + 32$. What range of values in Fahrenheit correspond to $-10 < C < 15$?

   Answer: $\boxed{14 < F < 59}$

3 Polynomial Inequalities

Method

Let's work through the process using $x^2 - x - 13 \geq 2x + 5$.

1. Add/subtract terms to get zero on the right-hand-side (RHS) of the inequality.

$$x^2 - 3x - 18 \geq 0$$

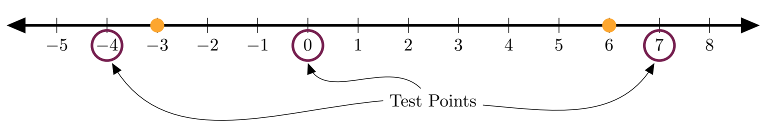
2. Factor the polynomial on the left-hand-side (LHS).

$$(x - 6)(x + 3) \geq 0$$

3. Set the factors on the LHS equal to zero and solve.

$$\begin{array}{ccc} x - 6 = 0 & \text{or} & x + 3 = 0 \\ x = 6 & \text{or} & x = -3 \end{array}$$

4. Plot the values from step 3 on a number line and pick *test points* on each side and between these numbers.



   - If the inequality is "<" or ">", plot them with open circles
   - If the inequality is "≤" or "≥", plot them with closed circles

5. Check to see if each interval is "good" or "bad" using the test points.

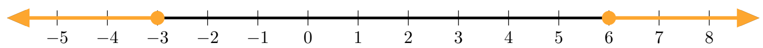
$$(x - 6)(x + 3) \geq 0$$

$x = -4 :\quad (-4 - 6)(-4 + 3) = (-10)(-1) = 10 \geq 0$ $\checkmark$

$x = 0 :\quad (0 - 6)(0 + 3) = (-6)(3) = -18 \geq 0$ $\times$

$x = 7: \quad (7-6)(7+3) = (1)(10) = 10 \geq 0 \quad \checkmark$

6. Fill in the "good" intervals and read the answer off the number line.



Answer: $\boxed{(-\infty, -3] \cup [6, \infty)}$

Example
Solve the polynomial inequality and write the answer in interval notation:

$4x^3 - 4x^2 - x + 1 < 0$

Answer: $\boxed{\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)}$

4 Rational Inequalities

Method
Let's work through the process using $\dfrac{3x+2}{2+x} < 2$.

1. Add/subtract terms to get zero on the right-hand-side (RHS) of the inequality and simplify to one fraction.

$$\frac{3x+2}{2+x} - 2 < 0$$
$$\frac{3x+2}{2+x} - \frac{2\cdot(2+x)}{1\cdot(2+x)} < 0$$
$$\frac{3x+2-4-2x}{2+x} < 0$$
$$\frac{x-2}{x+2} < 0$$

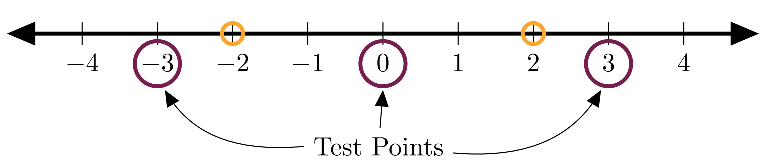
2. Factor the top and bottom of the fraction.

$$\frac{x-2}{x+2} < 0$$

3. Set all factors on top and bottom equal to zero and solve.

Top:	Bottom:
$x - 2 = 0$ $x = 2$	$x + 2 = 0$ $x = -2$

4. Plot the values from step 3 on a number line and pick *test points* on each side and between these numbers.



- For values from the bottom, *always* use open circles.
- On top values, plot them with open circles for "$<$" or "$>$".
- On top values, plot them with closed circles for "$\leq$" or "$\geq$".

5. Check to see if each interval is "good" or "bad" using the test points.

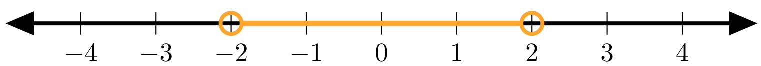
$$\frac{x-2}{x+2} < 0$$

$x = -3:\quad \frac{-3-2}{-3+2} = \frac{-5}{-1} = 5 < 0$ ✗

$x = 0:\quad \frac{0-2}{0+2} = \frac{-2}{2} = -1 < 0$ ✓

$x = 3:\quad \frac{3-2}{3+2} = \frac{1}{5} < 0$ ✗

6. Fill in the "good" intervals and read the answer off the number line.



Answer: $\boxed{(-2, 2)}$

Example
Solve the polynomial inequality and write the answer in interval notation:

$$\frac{(x-3)^2(x+1)}{(x-5)} \geq 0$$

Answer: $\boxed{(-\infty, -1] \cup \{3\} \cup (5, \infty)}$

5 Absolute Value Inequalities

Method
To solve, put the inequality into the standard form by isolating the absolute value on one side of the inequality.

Standard Form	$c > 0$ (normal case)	$c < 0$	$c = 0$
$\lvert X\rvert < c$	$-c < X < c$	no solution	no solution
$\lvert X\rvert \leq c$	$-c \leq X \leq c$	no solution	$X = 0$
$\lvert X\rvert > c$	$X < -c$ or $X > c$	all real numbers	all real numbers except $X = 0$
$\lvert X\rvert \geq c$	$X \leq -c$ or $X \geq c$	all real numbers	all real numbers

Examples
Solve the inequalities and write the answers in interval notation:

1. $4 - |2x + 3| \leq 3$

   Answer: $\boxed{(-\infty, -2] \cup (-2\infty)}$

2. $3|x + 4| + 2 > 1$

   Answer: $\boxed{(-\infty, \infty)}$