Section 10.3 Notes

Matrices and Systems of Linear Equations

1 Introduction

Definitions

- A *matrix* is a rectangular array of numbers.

  For example:
  $$\begin{bmatrix} 4 & -7 & \pi \\ -3 & 2 & 5 \end{bmatrix}$$

- The *dimension* of a matrix is the number of rows by the number of columns.

  For example, the matrix above is a 2×3 matrix.

- The *entries* of a matrix are the numbers inside the matrix.

  For example, in the matrix above, 4, -7, π, -3, 2, and 5 are the entries.

Augmented Matrix

- Every linear system can be written in a *standard form* - the terms with variables on the left hand side and the constant terms on the right.
- When systems are written in this form, there is an equivalent *augmented matrix* whose entries are the coefficients and constant terms.
- For example:

$$\begin{aligned} 2x + 3y - z &= 4 \\ 5x + y + z &= 0 \\ -x + 2z &= -1 \end{aligned} \qquad\qquad \begin{bmatrix} 2 & 3 & -1 & 4 \\ 5 & 1 & 1 & 0 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

- As a note, you might also see the augmented matrix written with a line. There's no difference between them.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 5 & 1 & 1 & 0 \\ -1 & 0 & 2 & -1 \end{array}\right]$$

2 Gaussian Elimination

Elementary Row Operations

To solve a system of equations using the augmented matrix, we will perform these three row operations on the matrix:

1. Switch any two rows.

$$\begin{bmatrix} 3 & 0 & -1 & 2 \\ 2 & 4 & 1 & -6 \\ -2 & 5 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & 5 & 0 & 3 \\ 2 & 4 & 1 & -6 \\ 3 & 0 & -1 & 2 \end{bmatrix}$$

2. Multiply the entries of a row by a nonzero number.

$$\begin{bmatrix} 3 & 0 & -1 & 2 \\ 2 & 4 & 1 & -6 \\ -2 & 5 & 0 & 3 \end{bmatrix} \xrightarrow{4R_2 \to R_2} \begin{bmatrix} 3 & 0 & -1 & 2 \\ 4 & 16 & 4 & -24 \\ -2 & 5 & 0 & 3 \end{bmatrix}$$

3. Multiply the entries of a row by a number and add it to another row.

$$\begin{bmatrix} 3 & 0 & -1 & 2 \\ 2 & 4 & 1 & -6 \\ -2 & 5 & 0 & 3 \end{bmatrix} \xrightarrow{2R_2 + R_3 \to R_3} \begin{bmatrix} 3 & 0 & -1 & 2 \\ 2 & 4 & 1 & -6 \\ 2 & 13 & 2 & -9 \end{bmatrix} \qquad \begin{array}{rrrrl} 4 & 8 & 2 & -12 & (2R_2) \\ -2 & 5 & 0 & 3 & (+R_3) \\ \hline 2 & 13 & 2 & -9 & \end{array}$$

Forms of an Augmented Matrix
When we are doing our row operations, our goal is to get the matrix into one of the following forms:

1. Row Echelon Form:
   - The first nonzero number in each row is a 1.
   - The first nonzero number in each row is farther right than the first nonzero number in the row above it.
   - Any row with only zeros is at the bottom of the matrix.
   - For example:

$$\begin{bmatrix} 1 & 3 & -2 & 7 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 6 & -11 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Reduced Row Echelon Form:
   - The same rules as Row Echelon form, but you also have to have all zeros above the leading 1 from each row.
   - For example:

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 0 & -11 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Strategy For Solving a System
We will be using row operations to get our matrix into either Row Echelon Form or Reduced Row Echelon Form. This process is called *Gaussian elimination* .

1. Figure out which of the two forms you're aiming for. Don't worry about the 1's until the last step - focus primarily on the location of the zeros.
   - If you want Row Echelon Form, this is your goal:

$$\begin{bmatrix} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$$

   - If you want Reduced Row Echelon Form, this is your goal:

$$\begin{bmatrix} \# & 0 & 0 & \# \\ 0 & \# & 0 & \# \\ 0 & 0 & \# & \# \end{bmatrix}$$

2. Work left to right, column by column to get the zeros where you want them for that column. You will primarily be using a combination of Row Operations 2 and 3 to get your zeros:

$$\begin{bmatrix} -2 & 4 & 1 & 3 \\ 3 & 5 & -2 & 4 \\ 7 & 8 & -5 & 11 \end{bmatrix} \xrightarrow{3R_1 + 2R_2 \to R_2} \begin{bmatrix} -2 & 4 & 1 & 3 \\ 0 & 22 & -1 & 17 \\ 7 & 8 & -5 & 11 \end{bmatrix} \qquad \begin{array}{rrrrl} -6 & 12 & 3 & 9 & (3R_1) \\ 6 & 10 & -4 & 8 & (+2R_2) \\ \hline 0 & 22 & -1 & 17 & \end{array}$$

   When you move on to the second and third columns, you need to be careful to not do something that will mess up the zeros you've already created.

To avoid this, make sure the two rows you use are:

- The same row that the entry you're changing comes from (this is absolutely mandatory - if you don't use this row your work is incorrect).
- The row with the same number as the column number for the entry your changing.

3. Once you have your zeros, you can get your 1's by dividing each row by it's leading nonzero number:

$$\begin{bmatrix} -2 & 1 & -8 & 1 \\ 0 & 4 & -1 & 13 \\ 0 & 0 & -5 & 10 \end{bmatrix} \xrightarrow[-\frac{1}{5}R_3 \to R_3]{\substack{-\frac{1}{2}R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2}} \begin{bmatrix} 1 & -1/2 & 4 & -1/2 \\ 0 & 1 & -1/4 & -13/4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

4. You should end up with one of three forms:

- When you have exactly one solution:

Row Echelon Form:

$$\begin{bmatrix} 1 & \# & \# & \# \\ 0 & 1 & \# & \# \\ 0 & 0 & 1 & \# \end{bmatrix}$$

Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{bmatrix}$$

- When you have no solution (if you see a row with all zeros except for the last entry *at any point*, simply stop because there is no solution):

$$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ 0 & 0 & 0 & a \end{bmatrix}$$

where a is any number except zero.

- When you have infinitely many solutions:

$$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. If you have a solution/solutions, write the matrices back in equation form, and if needed, back substitute to solve for the variables.

Examples

1.

$$\begin{aligned} 3r + 2s - 3t &= 10 \\ r - s - t &= -5 \\ r + 4s - t &= 20 \end{aligned}$$

$\boxed{(t, 5, t)}$

2.

$$\begin{aligned} 2x_1 + x_2 \qquad &= 7 \\ 2x_1 - x_2 + x_3 &= 6 \\ 3x_1 - 2x_2 + 4x_3 &= 11 \end{aligned}$$

$\boxed{(3, 1, 1)}$

3.

$$\begin{aligned} 2a + b - 2c &= 12 \\ -a - \tfrac{1}{2}b + c &= 6 \\ 3a + \tfrac{3}{2}b - 3c &= 18 \end{aligned}$$

$\boxed{\text{no solution}}$

4. A juice company sells three kinds of smoothies with the following recipes:

Smoothies	Mango Juice (oz)	Pineapple Juice (oz)	Orange Juice (oz)
Midnight Mango	8	3	3
Tropical Torrent	6	5	3
Pineapple Power	2	8	4

On a particular day, they used 820 oz of mango juice, 690 oz of pineapple juice, and 450 oz of orange juice. How many of each kind of smoothie was sold that day?

50 Midnight Mangos, 60 Tropical Torrents, 30 Pineapple Powers

5. John went to the candy store and spend \$50 on 30 pieces of candy. He bought a combination of bubble gum, chocolate bars, and lollipops, which cost \$2, \$3, and \$1 respectively. Find the general solution for how many of each type of candy he purchased and list three possible solutions.

General solution: $(40 - 2z, z - 10, z)$ where z is the number of lollipops.
Particular solutions:
11 lollipops, 1 chocolate bar, 18 bubble gums
12 lollipops, 2 chocolate bars, 16 bubble gums
13 lollipops, 3 chocolate bars, 14 bubble gums