Section 10.4 Notes

The Algebra of Matrices

1 Equality

Definition

- Two matrices are *equal* if they have the same dimensions and the corresponding entries in the same row and column are equal.
- For example:
  - $$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 13 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 13 \end{bmatrix}$$
  - $$\begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 5 & -6 \\ 4 & 3 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 5 & -6 \end{bmatrix}$$
  - $$\begin{bmatrix} 3 & 5 \\ -3 & -5 \end{bmatrix} \neq \begin{bmatrix} -3 & 5 \\ -3 & -5 \end{bmatrix}$$

Example

Solve for x and y.

$$\begin{bmatrix} x+y & 2-x \\ 4-x & x-y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$\boxed{x = 2,\ y = -1}$

2 Operations with Matrices

Addition and Subtraction

To add or subtract two matrices:

- Make sure the dimensions match - if not the sum/difference does not exist.
- Add or subtract the corresponding entries in the same row and column.

$$\begin{bmatrix} \# & \# & \# & \# \\ \# & a & \# & \# \\ \# & \# & \# & \# \end{bmatrix} + \begin{bmatrix} \# & \# & \# & \# \\ \# & b & \# & \# \\ \# & \# & \# & \# \end{bmatrix} = \begin{bmatrix} \# & \# & \# & \# \\ \# & a+b & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$$

$$\begin{bmatrix} \# & \# & \# & \# \\ \# & a & \# & \# \\ \# & \# & \# & \# \end{bmatrix} - \begin{bmatrix} \# & \# & \# & \# \\ \# & b & \# & \# \\ \# & \# & \# & \# \end{bmatrix} = \begin{bmatrix} \# & \# & \# & \# \\ \# & a-b & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$$

Multiplication by a Scalar

A *scalar* is a number (in contrast with a matrix).

To multiply a matrix by a scalar, just multiply all the elements by that scalar

$$c\begin{bmatrix} \# & \# & \# & \# \\ \# & a & \# & \# \\ \# & \# & \# & \# \end{bmatrix} = \begin{bmatrix} \# & \# & \# & \# \\ \# & ca & \# & \# \\ \# & \# & \# & \# \end{bmatrix}$$

Examples

Compute the following, if it exists:

1. $$\left[\begin{smallmatrix} 3 & 1 & -2 \\ 5 & -5 & -7 \end{smallmatrix}\right] + \left[\begin{smallmatrix} -1 & 0 & 22 \\ -4 & -10 & 2 \end{smallmatrix}\right]$$

   $\boxed{\left[\begin{smallmatrix} 2 & 1 & 20 \\ 1 & -15 & -5 \end{smallmatrix}\right]}$

2. $$3\left[\begin{smallmatrix} 4 & -1 & 2 \\ 0 & 1 & 0 \\ -10 & -4 & -3 \end{smallmatrix}\right]$$

   $\boxed{\left[\begin{smallmatrix} 12 & -3 & 6 \\ 0 & 3 & 0 \\ -30 & -12 & -9 \end{smallmatrix}\right]}$

3. $$\frac{1}{3}\left[\begin{smallmatrix} 0 \\ 3 \\ 2 \\ -6 \end{smallmatrix}\right] - \left[\begin{smallmatrix} -4 \\ 6 \\ -5 \\ -1 \end{smallmatrix}\right]$$

   $\boxed{\left[\begin{smallmatrix} 4 \\ -5 \\ 17/3 \\ -1 \end{smallmatrix}\right]}$

4. $$\left[\begin{smallmatrix} 2 & 1 & 0 \\ 3 & 4 & -13 \\ 7 & 1 & 6 \end{smallmatrix}\right] + 20\left[\begin{smallmatrix} -1 & 2 \\ 3 & -5 \\ 1 & 0 \end{smallmatrix}\right]$$

   $\boxed{\text{Does not exist}}$

5. Solve for the matrix X:

   $$2A = B - 3X,$$

   where

   $$A = \left[\begin{smallmatrix} 4 & 6 \\ 1 & 3 \end{smallmatrix}\right] \text{ and } B = \left[\begin{smallmatrix} 2 & 5 \\ 3 & 7 \end{smallmatrix}\right]$$

   $\boxed{X = \left[\begin{smallmatrix} -2 & -7/3 \\ 1/3 & 1/3 \end{smallmatrix}\right]}$

Matrix Multiplication

To multiply two matrices together:

- Check that the *inner dimensions* match - that is, make sure the number of columns of the first matrix is equal to the number of rows of the second matrix.

  $$\begin{bmatrix} \# & \# & \# & \# \\ a_1 & a_2 & a_3 & a_4 \\ \# & \# & \# & \# \end{bmatrix}\begin{bmatrix} \# & \# & \# & b_1 & \# \\ \# & \# & \# & b_2 & \# \\ \# & \# & \# & b_3 & \# \\ \# & \# & \# & b_4 & \# \end{bmatrix} = \begin{bmatrix} \# & \# & \# & \# & \# \\ \# & \# & \# & c & \# \\ \# & \# & \# & \# & \# \end{bmatrix}$$

  where $\boxed{c = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4}$

- The dimensions of the final matrix are determined by the *outer dimensions* - that is, the number of rows of the first matrix and the number of columns of the second.

Examples

Compute the following, if they exist:

1. $$\begin{bmatrix} 4 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 & -1 \\ 1 & 7 & 6 \end{bmatrix}$$

   $\boxed{\begin{bmatrix} 23 & 21 & 14 \\ 8 & -14 & -14 \end{bmatrix}}$

2. AB and BA where $A = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \end{bmatrix}$.

   $\boxed{AB \text{ does not exist}}$

   $\boxed{BA = \begin{bmatrix} -5 & 7 & 7 \end{bmatrix}}$

3. AB and BA where $A = \begin{bmatrix} 3 & 6 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -8 \\ -3 & 1 \end{bmatrix}$.

   $\boxed{AB = \begin{bmatrix} -12 & -18 \\ -8 & 10 \end{bmatrix}}$

   $\boxed{BA = \begin{bmatrix} 14 & 4 \\ -4 & -16 \end{bmatrix}}$