Section 10.5 Notes

Inverses of Matrices and Matrix Equations

1 Identity Matrices

Definition

- The $n \times n$ identity matrix I_n is the matrix with 1's on the diagonal and 0's in every other spot.
- For example:
  - $I_1 = \begin{bmatrix} 1 \end{bmatrix}$
  - $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
  - $I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
- Frequently, if the dimension of the identity matrix is clear by the context of the problem, the subscript will be dropped.

Identiy Matrix and Multiplication

- When dealing with matrix multiplication, the identity matrix performs the same function as the number 1 does:
  - $1 \cdot x = x$
  - $x \cdot 1 = x$
- For example:
  - $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{2+0} & \underline{3+0} & \underline{-1+0} \\ \underline{0+5} & \underline{0+0} & \underline{0+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 1 \end{bmatrix}$$
  - $$\begin{aligned} \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} \underline{2+0+0} & \underline{0+3+0} & \underline{0+0-1} \\ \underline{5+0+0} & \underline{0+0+0} & \underline{0+0+1} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 1 \end{bmatrix} \end{aligned}$$
- In other words, if A is an $m \times n$ matrix:

  - $I_m A = A$
  - $A I_n = A$

2 Matrix Inverses

Definition

- When dealing with real numbers, $\frac{1}{x}$ is the multiplicative *inverse* of x, which means it has the property:
  - $x \cdot \frac{1}{x} = 1$
- With matrices, we have a very similar idea. If A is an $n \times n$ matrix, then A^{-1} (read A inverse) is another $n \times n$ matrix where:
  - $AA^{-1} = I_n$
  - $A^{-1}A = I_n$

Example

Show that $A = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ are inverses.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding Matrix Inverses

Let's work through finding A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

1. Write out the matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -5 & -13 & 0 & 1 \end{array}\right]$$

2. Use row operations to transform the matrix into $\left[\begin{array}{c|c} I_n & B \end{array}\right]$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -5 & -13 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & -1 & 5 & 2 \end{array}\right] \rightarrow$$

$$\begin{array}{ccccl} 10 & 25 & 5 & 0 & (5R_1) \\ -10 & -26 & 0 & 2 & (+2R_2) \\ \hline 0 & -1 & 5 & 2 & (\text{New } R_2) \end{array} \qquad \begin{array}{ccccl} 2 & 5 & 1 & 0 & (R_1) \\ 0 & -5 & 25 & 10 & (+5R_2) \\ \hline 2 & 0 & 26 & 10 & (\text{New } R_1) \end{array}$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 26 & 10 \\ 0 & -1 & 5 & 2 \end{array}\right] \xrightarrow[]{\substack{\frac{1}{2}R_1 \to R_1 \\ -R_2 \to R_2}} \left[\begin{array}{cc|cc} 1 & 0 & 13 & 5 \\ 0 & 1 & -5 & -2 \end{array}\right]$$

3. Once in this form, $A^{-1} = B$.

$$A^{-1} = \begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}$$

4. If, at any point in step 2, you obtain all zeros in a row on the left side, this means the matrix is not invertible:

$$\left[\begin{array}{ccc|ccc} \# & \# & \# & \# & \# & \# \\ 0 & 0 & 0 & \# & \# & \# \\ \# & \# & \# & \# & \# & \# \end{array}\right]$$

Examples

Find the inverse of the following matrices:

1. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 3 & -2 & -5 \\ -1 & 1 & 1 \\ -3 & 2 & 6 \end{bmatrix}}$$

2. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$\boxed{\text{No inverse}}$

3 Solving Systems of Equations with Matrix Inverses

Definitions

- The *coefficient matrix* of a linear system is the matrix containing the coefficients of the variables.

  For example:

$$\begin{aligned} 2x + 3y - z &= 4 \\ 5x + y + z &= 0 \\ -x + 2z &= -1 \end{aligned}$$

  The coefficient matrix for this system is

$$\begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

- We can also create a column matrix with the constant terms from the right-hand-side:

$$\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

Matrix Equation

- The previous system can be rewritten as a matrix equation:

$$\begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

- Notice that this happens because of how matrix multiplication works:

$$\begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \underline{2x + 3y - z} \\ \underline{5x + y + z} \\ \underline{-x + 2z} \end{bmatrix}$$

$$\begin{bmatrix} 2x + 3y - z \\ 5x + y + z \\ -x + 2z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

  Remember, when two matrices are equal, that just means each corresponding entry is equal:

$$\begin{aligned} 2x + 3y - z &= 4 \\ 5x + y + z &= 0 \\ -x + 2z &= -1 \end{aligned}$$

- This means, if A is the coefficient matrix, B is the column matrix containing the constant terms from the right-hand-side, and X is the column matrix containing the variables, a linear system can be written in the following form:

$$AX = B$$

Solving Systems of Equations

If A is an inverible square matrix, then the matrix equation $AX = B$ can be solved with the formula:

$$\boxed{X = A^{-1}B}$$

Reasoning:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$I_nX = A^{-1}B$$

$$X = A^{-1}B$$

Examples

Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix.

$$\begin{aligned} -2y + 2z &= 4 \\ 3x + y + 3z &= 1 \\ x - 2y + 3z &= -2 \end{aligned}$$

$$\boxed{(-27, 19, 21)}$$