Section 10.6 Notes

Determinants and Cramer's Rule

1 Determinants

Definition

- The *determinant* of a square matrix is a number that's computed using the entries of that matrix.
  - *Only* square matrices have determinants.
  - If a problem asks you to find the determinant of a non-square matrix, the answer is simply "does not exist".
- There are a couple notations you might see for determinants:
  - $|A|$ - when using this notation, the square brackets are dropped.
  - $\det(A)$

Determinant of a 2×2 Matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example:

$$\begin{aligned} \begin{vmatrix} -4 & 3 \\ -1 & 5 \end{vmatrix} &= (-4)(5) - (3)(-1) \\ &= -20 + 3 \\ &= -17 \end{aligned}$$

Determinant of an $n \times n$ Matrix, $n > 2$

Let's find the determinant of $\begin{vmatrix} 4 & -1 & 3 \\ 2 & 0 & 4 \\ -4 & 11 & 6 \end{vmatrix}$

1. Determine the "sign grid" for whatever sized matrix you're working with:

   - $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$
   - $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$
   - $\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$

2. Pick a row or column from the matrix to expand on - preferably one with as many zeros as possible.

$$\begin{vmatrix} 4 & -1 & 3 \\ 2 & 0 & 4 \\ -4 & 11 & 6 \end{vmatrix}$$

3. Write out the numbers from that row/column. Leave lots of space after each number, and also apply the signs from the "sign grid" for that row or column.

$$-(-1) \qquad\qquad +(0) \qquad\qquad -(11)$$

4. Multiply each number by the determinant you'd get by eliminating the row and column that the number came from:

$$\begin{vmatrix} 4 & -1 & 3 \\ 2 & 0 & 4 \\ -4 & 11 & 6 \end{vmatrix} \qquad\qquad \begin{vmatrix} 4 & -1 & 3 \\ 2 & 0 & 4 \\ -4 & 11 & 6 \end{vmatrix} \qquad\qquad \begin{vmatrix} 4 & -1 & 3 \\ 2 & 0 & 4 \\ -4 & 11 & 6 \end{vmatrix}$$

$$-(-1)\begin{vmatrix} 2 & 4 \\ -4 & 6 \end{vmatrix} + (0)\begin{vmatrix} 4 & 3 \\ -4 & 6 \end{vmatrix} - (11)\begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix}$$

5. Simplify.

$$1 \cdot \begin{vmatrix} 2 & 4 \\ -4 & 6 \end{vmatrix} + \cancel{0 \cdot \begin{vmatrix} 4 & 3 \\ -4 & 6 \end{vmatrix}} - 11 \cdot \begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix}$$

$$\begin{aligned} &= 1\left((2)(6) - (4)(-4)\right) - 11\left((4)(4) - (3)(2)\right) \\ &= 1(12 + 16) - 11(16 - 6) \\ &= 1(28) - 11(10) \\ &= 28 - 110 \\ &= -82 \end{aligned}$$

$$\boxed{-82}$$

Examples

Find the determinants.

1. $\begin{vmatrix} 5 & 3 & -1 \\ -2 & 1 & 3 \\ 5 & 1 & -1 \end{vmatrix}$

$\boxed{26}$

2. $\begin{vmatrix} 2 & 0 & 3 & -4 \\ 0 & 5 & -1 & 2 \\ 3 & 0 & 6 & 1 \\ -2 & 1 & 0 & 4 \end{vmatrix}$

$\boxed{-230}$

2 Cramer's Rule

Formula
Suppose you have a system of linear equations with variables $x_1, x_2, \ldots, x_n$. If A is the coefficient matrix and B is the column matrix from the right-hand-side, then

$$\boxed{x_k = \frac{\det(A_{x_k})}{\det(A)}}$$

where A_{x_k} is the matrix A with the kth column replaced by B.

Example

Solve the system using Cramer's Rule.

$$\begin{aligned} 4x - 3y &= 13 \\ -2x + y &= -5 \end{aligned}$$

$\boxed{(1, -3)}$

3 Other Uses of Determinants

Invertibility Criterion

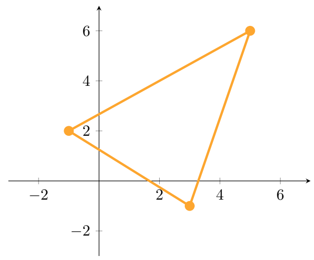
- If $\det(A) = 0$, then A is not inverible (A has no inverse).
- If $\det(A) \neq 0$, then A is invertible (A has an inverse).

Area of a Triangle
If the three vertices of a triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3), then you can find the area of the triangle in the following way:

- First compute
$$s = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
- The area is given by
$$A = \left|\frac{s}{2}\right|$$
(Here the vertical lines are absolute value symbols, not determinants.)

Example

Find the area of the triangle with vertices at $(-1, 2)$, $(3, -1)$, and $(5, 6)$.



17