

## Section 10.7 Notes

### Partial Fractions

#### Definition

- *Partial fraction decomposition* is essentially undoing fraction addition.
- For example, to add two fractions together:

$$\begin{aligned}\frac{4}{x+6} + \frac{3}{x-4} &= \frac{4(x-4)}{(x+6)(x-4)} + \frac{3(x+6)}{(x-4)(x+6)} \\ &= \frac{4x-16+3x+18}{(x+6)(x-4)} \\ &= \frac{7x+2}{(x+6)(x-4)}\end{aligned}$$

- Now, a question will give us  $\frac{7x+2}{(x+6)(x-4)}$  and ask us to find the partial fraction decomposition. The answer here would be  $\frac{4}{x+6} + \frac{3}{x-4}$ .

#### Method

1. If the degree on top is greater than or equal to the degree on the bottom, perform polynomial long division. The rest of the steps are only referring to the fractional part, so you'll want to ignore the quotient until the very end.
2. Factor the denominator.
3. Write out the form of the decomposition. This depends on two features of the factors in the denominator.

Type of Factor in the Denominator:

- If the factor is linear, then the numerator will be a constant.

Example:

$$\frac{2x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

- If the factor is quadratic, then the numerator will be linear.

Example:

$$\frac{6}{(x^2-x-1)(x+5)} = \frac{Ax+B}{x^2-x-1} + \frac{C}{x+5}$$

- If the factor is cubic, then the numerator will be quadratic.

Example:

$$\frac{3x+5}{(x-4)(x^3-7x+1)} = \frac{A}{x-4} + \frac{Bx^2+Cx+D}{x^3-7x+1}$$

Multiplicity of the Factor in the Denominator:

- If the factor has multiplicity 1, then there will be one fraction for that factor.

Example:

$$\frac{2x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

- If the factor has multiplicity 2, then there will be two fractions for that factor.

Example:

$$\frac{2x}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

- If the factor has multiplicity 3, then there will be three fractions for that factor.

Example:

$$\frac{2x}{(x-1)^3(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+3}$$

4. Recombine the decomposition into a single fraction. Set the numerators equal and solve for the unknown variables ( $A$ ,  $B$ ,  $C$ , etc). There are two possible ways of completing this step.
5. Plug the values back into the decomposed form. If you had to do long division in step 1, don't forget to add the quotient back in.

#### Option 1: Plug in different numbers for $x$

Let's find the decomposition of  $\frac{5x}{x^2 - 2x - 3}$ .

- Set up the decomposition and recombine the fractions:

$$\begin{aligned}\frac{5x}{x^2 - 2x - 3} &= \frac{5x}{(x-3)(x+1)} \\ &= \boxed{\frac{A}{x-3} + \frac{B}{x+1}} \\ &= \frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x-3)(x+1)} \\ &= \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}\end{aligned}$$

- Set the numerators equal:

$$A(x+1) + B(x-3) = 5x$$

- Plug in  $x = -1$  and solve:

$$\begin{aligned}A(0) + B(-4) &= 5(-1) \\ -4B &= -5 \\ B &= \frac{5}{4}\end{aligned}$$

- Plug in  $x = 3$  and solve:

$$\begin{aligned}A(4) + B(0) &= 5(3) \\ 4A &= 15 \\ A &= \frac{15}{4}\end{aligned}$$

- Plug the variables back in to the decomposed form:

$$\frac{15/4}{x-3} + \frac{5/4}{x+1}$$

$$\frac{15}{4(x-3)} + \frac{5}{4(x+1)}$$

## Option 2: Create a System of Equations

Let's find the decomposition of  $\frac{x^3 + 4x^2 + 7x + 1}{x^2 + 3x + 2}$ .

- Perform long division.

$$\begin{array}{r} x+1 \\ x^2+3x+2 \overline{) x^3+4x^2+7x+1} \\ \underline{-x^3-3x^2-2x} \phantom{+1} \\ x^2+5x+1 \\ \underline{-x^2-3x-2} \\ 2x-1 \end{array}$$

The result is  $x + 1 + \frac{2x-1}{x^2+3x+2}$ , but for now we'll just use the  $\frac{2x-1}{x^2+3x+2}$ .

- Set up the decomposition and recombine the fractions:

$$\begin{aligned} \frac{2x-1}{x^2+3x+2} &= \frac{2x-1}{(x+2)(x+1)} \\ &= \frac{A}{x+2} + \frac{B}{x+1} \\ &= \frac{A(x+1)}{(x+2)(x+1)} + \frac{B(x+2)}{(x+2)(x+1)} \\ &= \frac{Ax + A + Bx + 2B}{(x+2)(x+1)} \end{aligned}$$

- Set the numerators equal and "combine like terms."

$$\begin{aligned} Ax + A + Bx + 2B &= 2x - 1 \\ Ax + Bx + A + 2B &= 2x - 1 \\ (A + B)x + (A + 2B) &= 2x - 1 \end{aligned}$$

- Create a system of equations by setting the coefficients of like terms equal.

$$\begin{aligned} A + B &= 2 \\ A + 2B &= -1 \end{aligned}$$

- Solve the system of equations:

$$\begin{array}{rcl} -A & -B & = -2 \\ A & +2B & = -1 \\ \hline & B & = -3 \end{array} \qquad \begin{array}{l} A + (-3) = 2 \\ A = 5 \end{array}$$

- Plug the variables back in to the decomposed form:

$$x + 1 + \frac{5}{x + 2} - \frac{3}{x + 1}$$

### Examples

Find the partial fraction decomposition of each fraction:

1.  $\frac{7x + 1}{(3x + 1)^2(x - 1)}$

$$-\frac{3}{2(3x + 1)} + \frac{1}{(3x + 1)^2} + \frac{1}{2(x - 1)}$$

2.  $\frac{2x + 7}{(x - 1)(x^2 + x + 1)}$

$$\frac{3}{x - 1} - \frac{3x + 4}{x^2 + x + 1}$$

3.  $\frac{x^3 - 3x^2 + 3x - 15}{x^4 + 3x^2}$

$$\frac{1}{x} - \frac{5}{x^2} + \frac{2}{x^2 + 3}$$