

Section 12.6 Notes

The Binomial Theorem

1 Formulas

Factorial

$$n! = \begin{cases} n(n-1) \dots (2)(1) & \text{if } n = 2, 3, 4, \dots \\ 1 & \text{if } n = 0 \text{ or } 1 \end{cases}$$

For example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Examples

1. Compute $6!$.

720

2. Compute $\frac{10!}{8!}$.

90

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The binomial coefficient is read as “ n choose k ”. The number on the bottom is *never* bigger than the number on top.

For example:

$$\begin{aligned} \binom{5}{3} &= \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} \\ &= \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})(2 \cdot 1)} \\ &= \frac{20}{2} = 10 \end{aligned}$$

Example

Compute $\binom{10}{1} + \binom{10}{9}$.

20

2 Pascal's Triangle

Construction of Pascal's Triangle

To generate Pascal's Triangle, you start with a 1 at the top. Then, the numbers for each subsequent row is found by adding the two numbers above it. You can imagine that everything "outside" of the triangle is a zero.

$$\begin{array}{ccccccc}
 & & & & 1 & & & & \\
 & & & & & 1 & & 1 & \\
 & & & 1 & & 2 & & 1 & \\
 & & 1 & & 3 & & 3 & & 1 \\
 & 1 & & 4 & & 6 & & 4 & & 1 \\
 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
 \end{array}$$

Relationship to Binomial Coefficients

It turns out, that you can use Pascal's Triangle to compute the binomial coefficients rather than using the formula.

$$\begin{array}{cccccccc}
 & & & & \binom{0}{0} = 1 & & & \\
 & & & \binom{1}{0} = 1 & & \binom{1}{1} = 1 & & \\
 & & \binom{2}{0} = 1 & & \binom{2}{1} = 2 & & \binom{2}{2} = 1 & \\
 & \binom{3}{0} = 1 & & \binom{3}{1} = 3 & & \binom{3}{2} = 3 & & \binom{3}{3} = 1 \\
 & \binom{4}{0} = 1 & & \binom{4}{1} = 4 & & \binom{4}{2} = 6 & & \binom{4}{3} = 4 & & \binom{4}{4} = 1 \\
 \binom{5}{0} = 1 & & \binom{5}{1} = 5 & & \binom{5}{2} = 10 & & \binom{5}{3} = 10 & & \binom{5}{4} = 5 & & \binom{5}{5} = 1 \\
 \binom{6}{0} = 1 & & \binom{6}{1} = 6 & & \binom{6}{2} = 15 & & \binom{6}{3} = 20 & & \binom{6}{4} = 15 & & \binom{6}{5} = 6 & & \binom{6}{6} = 1
 \end{array}$$

Each row of the triangle gives you all the coefficients for a particular n in $\binom{n}{k}$:

$$\begin{array}{lcl}
 n = 0: & & 1 \\
 n = 1: & & 1 \quad 1 \\
 n = 2: & & 1 \quad 2 \quad 1 \\
 n = 3: & & 1 \quad 3 \quad 3 \quad 1 \\
 n = 4: & & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 n = 5: & & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 n = 6: & & 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
 \end{array}$$

3 Multiplying Binomial Expressions

The Binomial Theorem

The Binomial Theorem gives a formula for multiplying out binomial expressions:

$$(A + B)^n = \binom{n}{0}A^n + \binom{n}{1}A^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \dots \\ + \binom{n}{n-2}A^2B^{n-2} + \binom{n}{n-1}AB^{n-1} + \binom{n}{n}B^n$$

Examples

1. Expand $(x + y)^3$, and compute the binomial coefficients using the formula.

$$x^3 + 3x^2y + 3xy^2 + y^3$$

2. Expand $(2r + s^2)^4$, using Pascal's Triangle to find the coefficients.

$$16r^2 + 32r^3s^2 + 24r^2s^4 + 8rs^6 + s^8$$

3. Factor using the Binomial Theorem: $(x + 4)^6 + 6(x + 4)^5 + 15(x + 4)^4 + 20(x + 4)^3 + 15(x + 4)^2 + 6(x + 4) + 1$.

$$(x + 5)^6$$

Finding a Single Term

The term that contains $A^{n-k}B^k$ in $(A + B)^n$ is

$$\binom{n}{k}A^{n-k}B^k$$

Examples

1. Find the coefficient of x^2y^3 in $(5x - 2y)^5$.

$$-2000$$

2. Find the tenth term in the expansion of $(2x + \frac{y}{2})^{14}$

$$\frac{1001}{8}x^5y^9$$