Section 2.6 Notes

Combining Functions

1 Combining Functions Algebraically

Adding two Functions

- If f and g are two functions,
$$(f+g)(x) = f(x) + g(x)$$
- If the domain of f is A and the domain of g is B, then the domain of $f+g$ is $A \cap B$ (recall the intersection of two sets is where they overlap).
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x-2$:
$$(f+g)(x) = \sqrt{x+1} + x - 2$$
The domain of $f+g$ is $x \geq -1$, since the domain of f is $x \geq -1$ and the domain of g is all real numbers.

Subtracting two Functions

- If f and g are two functions,
$$(f+g)(x) = f(x) - g(x)$$
- If the domain of f is A and the domain of g is B, then the domain of $f-g$ is $A \cap B$.
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x-2$:
$$(f-g)(x) = \sqrt{x+1} - (x-2) = \sqrt{x+1} - x + 2$$
The domain of $f-g$ is $x \geq -1$, since the domain of f is $x \geq -1$ and the domain of g is all real numbers.

Multiplying two Functions

- If f and g are two functions,
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
- If the domain of f is A and the domain of g is B, then the domain of $f \cdot g$ is $A \cap B$.
- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x-2$:
$$(f \cdot g)(x) = \sqrt{x+1} \cdot (x-2) = (x-2)\sqrt{x+1}$$
The domain of $f-g$ is $x \geq -1$, since the domain of f is $x \geq -1$ and the domain of g is all real numbers.

Dividing two Functions

- If f and g are two functions,
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

- If the domain of f is A and the domain of g is B, then the domain of $\left(\frac{f}{g}\right)(x)$ is $A \cap B$, but also removing any points where $g(x) = 0$.

- For example, suppose $f(x) = \sqrt{x+1}$ and $g(x) = x - 2$:
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+1}}{x-2}$$
The domain of $f - g$ is $[-1, 2) \cup (2, \infty)$, since the domain of f is $x \geq -1$, the domain of g is all real numbers, but $g(x) = 0$ when $x = 2$.

Practical Approach to Finding the Domains

- Find the formula for the function (and simplify if needed).

- Look at the *unsimplified* formula. Apply the same rules we've used before for finding domains:

  - If the expression contains $\boxed{\frac{A}{B}}$, then $\boxed{B \neq 0}$.
  - If the expression contains $\boxed{\sqrt{B}}$, then $\boxed{B \geq 0}$.

  - If the expression contains $\boxed{\frac{A}{\sqrt{B}}}$, then $\boxed{B > 0}$. This rule *only* works when the *entire* denominator is under a square root.

- Also notice that the domains for $f + g$, $f - g$ and $f \cdot g$ will always be the same.

Examples
Find the new functions and determine their domains.

1. $f(x) = \frac{1}{x-2}$, $g(x) = \sqrt{3x-1}$; find $(f+g)(x)$

$$(f+g)(x) = \frac{1}{x-2} + \sqrt{3x-1}$$
Domain: $\left[\frac{1}{3}, 2\right) \cup (2, \infty)$

2. $f(x) = 2x - 3$, $g(x) = \frac{1}{\sqrt{2x^2+x}}$; find $(f \cdot g)(x)$

$$(f \cdot g)(x) = \frac{2x-3}{\sqrt{2x^2+x}}$$

Domain: $\left(-\infty, -\frac{1}{2}\right) \cup (0, \infty)$

Find the new functions and determine their domains.

3. $f(x) = x^2 - 1$, $g(x) = x - 1$; find $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = x + 1$$

Domain: $(-\infty, 1) \cup (1, \infty)$

2 Function Composition

Definition

- *Composition* is the process of plugging one function into another:

$$(f \circ g)(x) = f\left(g(x)\right)$$

- For example: $f(x) = 2x - 1$ and $g(x) = x + 5$:

  $(f \circ g)(x) = f(x+5) = 2(x+5) - 1 = 2x + 10 - 1 = 2x + 9$

- Multiple Compositions: $(f \circ g \circ h)(x) = f\left(g\left(h(x)\right)\right)$
- For example, $f(x) = 4x + 5$, $g(x) = x^2$, $h(x) = \sqrt{x+2}$

$$\begin{aligned} g(h(x)) &= g(\sqrt{x+2}) = (\sqrt{x+2})^2 = x + 2 \\ (f \circ g \circ h) &= f(g(h(x))) = f(x+2) = 4(x+2) + 5 = 4x + 13 \end{aligned}$$

- The domain can be found by looking at the *unsimplified* formula, and using our usual rules for finding domains.

Examples

1. If $f(x) = 2x^2$ and $g(x) = \sqrt{2x-4}$, find $f \circ g$, $g \circ f$ and their domains.

$$(f \circ g)(x) = 4x - 8$$

Domain: $[2, \infty)$

$$(g \circ f)(x) = \sqrt{4x^2 - 4}$$

Domain: $(-\infty, -1] \cup [1, \infty)$

2. If $f(x) = \frac{x-1}{2x+1}$, find $f \circ f$ and its domain.

$$(f \circ f)(x) = \frac{-x-2}{4x-1}$$

Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)$