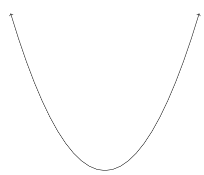
Section 3.1 Notes

Quadratic Functions and Models

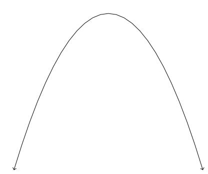
1 Introduction

Definitions

- A *quadratic function* is a function with the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.
- The graphs of quadratic functions are all *parabolas* - informally, they have a "bowl" or a "U" shape, either upside down or right-side up.



$a > 0$
We say the parabola *opens up*.



$a < 0$
We say the parabola *opens down*.

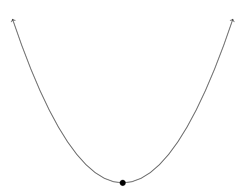
2 Standard Form of a Quadratic

Definitions

- Every quadratic can be written in *standard form* (also called *vertex form*):

$$f(x) = a(x - h)^2 + k$$

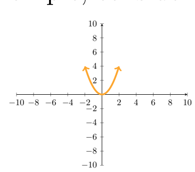
- In this form, (h, k) is the *vertex* of the parabola (and a still determines if the parabola opens up or down):



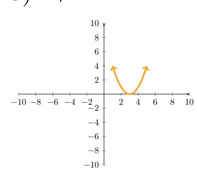
(h, k)

Graphing using Transformations

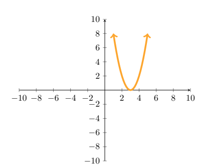
- By writing a quadratic in the standard form, we can see that every quadratic is just $y = x^2$ with some transformations done to it.
- For example, consider $y = -2(x - 3)^2 + 4$:



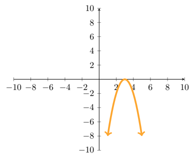
$y = x^2$

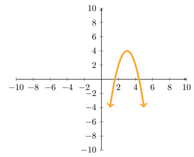


$y = (x - 3)^2$



$y = 2(x - 3)^2$





Putting Quadratic Functions into Standard Form
Method 1: Completing the Square
Let's do $f(x) = -3x^2 + 5x + 1$ as an example.

1. Factor out the coefficient of x^2 from the x^2 and x terms.

$$f(x) = -3\left(x^2 - \frac{5}{3}x\right) + 1$$

2. Perform the usual scratch work:

$$-\tfrac{5}{3} \cdot \tfrac{1}{2} = -\tfrac{5}{6} \qquad\qquad \left(-\tfrac{5}{6}\right)^2 = \tfrac{25}{36}$$

3. Add *and* subtract the second number inside the parenthesis:

$$f(x) = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36} - \frac{25}{36}\right) + 1$$

4. Regroup the first three terms inside the parenthesis and distribute the outside coefficient.

$$\begin{aligned} f(x) &= -3\left(\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) - \frac{25}{36}\right) + 1 \\ f(x) &= -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) + 3 \cdot \frac{25}{36} + 1 \\ f(x) &= -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) + \frac{25}{12} + \frac{12}{12} \\ f(x) &= -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) + \frac{37}{12} \end{aligned}$$

5. Factor inside the parenthesis using the first number from the scratch work:

$$f(x) = -3\left(x - \frac{5}{6}\right)^2 + \frac{37}{12}$$

Method 2: Formula
The vertex of the parabola $f(x) = ax^2 + bx + c$ is given by:

$$\begin{aligned} h &= -\frac{b}{2a} \\ k &= f(h) \end{aligned}$$

1. Calculate h and k using the formulas, and get a from the original equation.

2. Plug these into $f(x) = a(x-h)^2 + k$

Examples
Write the following quadratic functions into standard form:

1. $f(x) = x^2 - 4x + 5$

   $\boxed{f(x) = (x-2)^2 + 1}$

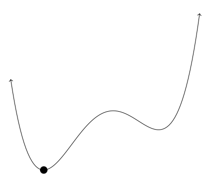
2. $f(x) = -3x^2 + 4x + 1$

   $\boxed{f(x) = -3\left(x - \frac{2}{3}\right) + \frac{7}{3}}$

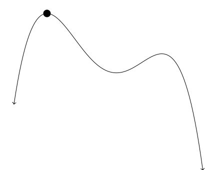
3 Maximums and Minimums of Quadratics

Absolute Maximums and Minimums

- An *absolute maximum* or *maximum* is a point on the graph that is higher than every other point.
- An *absolute minimum* or *minimum* is a point on the graph that is lower than every other point.

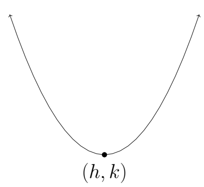


This has an absolute minimum but no absolute maximum.

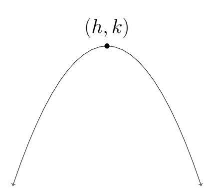


This has an absolute maximum but no absolute minimum.

Maximums and Minimums on a Quadratic
Every quadratic has either a maximum or a minimum at its vertex.



If $a > 0$, it has a minimum.



If $a < 0$, it has a maximum.

Examples

1. Suppose a rectangle has a perimeter of 120 inches. What is the largest area the rectangle can have?

   $\boxed{900 \text{ in}^2}$

2. A soft-drink vendor at a popular beach analyzes his sales records and finds if he sells x cans of soda in one day, his profit (in dollars) is given by

$$P(x) = -0.001x^2 + 3x - 1800$$

What is his maximum profit and how many cans must he sell for maximum profit?

Maximum profit is \$450 and he needs to sell 1500 cans.

4 Graphing Quadratics without Transformations

Method

1. Find the vertex of the parabola.
2. Find the x and y intercepts.
3. Plot all of the above points and connect them with a parabola.

Example

Graph $f(x) = 2x^2 - 7x - 4$.