Section 3.2 Notes

Polynomial Functions and Their Graphs

1 Introduction

Definitions

- A *polynomial function* is a function with the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where

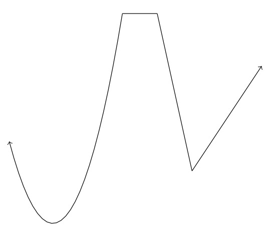
  - a_n, $a_{n-1}, \ldots, a_1$, and a_0 are numbers, and are known as *coefficients*
  - a_n is the *leading coefficient* (and should be $\neq 0$)
  - n is the *degree*
  - a_0 is the *constant term*

- For example, $P(x) = 3x^2 - 2$ and $P(x) = x^5 + 4x^4 - x$ are polynomial functions.

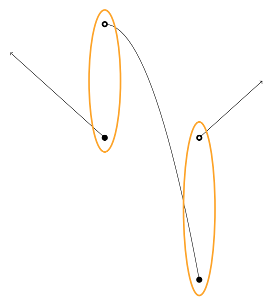
2 Properties of Polynomial Graphs

Continuity

The graphs of polynomial functions are *continuous*: there are no breaks in the graph, and can be drawn without lifting the pen or pencil.



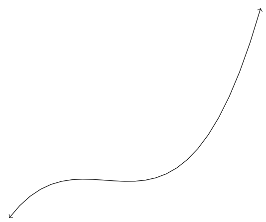
This is continuous.



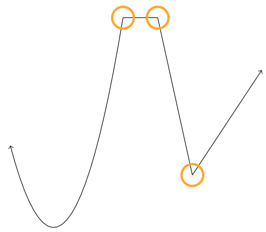
This is *not* continuous.

Smoothness

The graphs of polynomial functions are *smooth*: there are no sharp corners/cusps.



This is smooth.

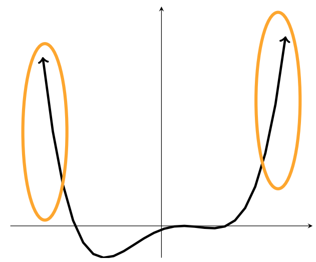


This is not smooth.

3 End Behavior

Definition

- The *end behavior* of a polynomial is a description of what happens to the y-values as you plug in extremely large (positive or negative) x-values.
- In terms of the graph, it's useful to think of the end behavior as the the appearance of the graph on the two sides outside of all of the x-intercepts.



Example

We'll determine the end behavior of $f(x) = -3x^3 - 2x$ by plugging in large values for x.

Large Positive Numbers:

x	$f(x)$
100	-3,000,200
1000	-3,000,002,000
10000	-3,000,000,020,000

Large Negative Numbers:

x	$f(x)$
-100	3,000,200
-1000	3,000,002,000
-10000	3,000,000,020,000

- We see that as x gets large in the positive direction, y gets large in the negative direction, and as x get's large in the negative direction, y gets large in the positive direction.
- We can use an arrow notation to write that a little more nicely:

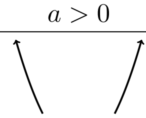
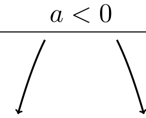
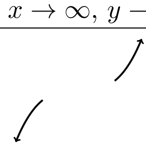
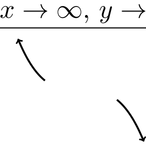
  As $x \to \infty$, $y \to -\infty$.

  As $x \to -\infty$, $y \to \infty$.
- If you work through the computations, you can see that the y-value is almost completely determined by the $-3x^3$ term.

Determining the End Behavior

To figure out the end behavior, the only thing that matters is the leading term.

Suppose $\mathbf{ax^n}$ is the leading term:

	$a > 0$	$a < 0$
n is even	 As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to \infty$	 As $x \to -\infty$, $y \to -\infty$ As $x \to \infty$, $y \to -\infty$
n is odd	 As $x \to -\infty$, $y \to -\infty$ As $x \to \infty$, $y \to \infty$	 As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to -\infty$

Examples
Determine the end behavior of the polynomial.

1. $P(x) = -2x^5 + x^2 - 1$

   As $x \to -\infty$, $y \to \infty$.
   As $x \to \infty$, $y \to -\infty$.

2. $P(x) = 3x^4 - x^3 + x^2 + 5x + 8$

   As $x \to -\infty$, $y \to \infty$.
   As $x \to \infty$, $y \to \infty$.

4 Multiplicity of Zeros

Zeros of a Polynomial

- Recall that *zeros* (or *roots*, *x-intercepts*) are points where the graph crosses the x-axis, and are the values of x when $y = 0$.
- To find the zeros of a polynomial, you must use the factored form of the polynomial.
- Plug in 0 for $f(x)$, then set each factor equal to zero and solve.

  For example, we can find the zeros of $P(x) = x^3 - 3x^2 - 4x + 12$:

  $0 = (x^3 - 3x^2) + (-4x + 12)$

  $0 = x^2(x - 3) - 4(x - 3)$

  $0 = (x - 3)(x^2 - 4)$

  $0 = (x - 3)(x - 2)(x + 2)$

  $x - 3 = 0$, $x - 2 = 0$, $x + 2 = 0$

  $x = 3$, $x = 2$, $x = -2$

Definition of Multiplicity

- The *multiplicity* of a zero a is the number of linear factors that, when set equal to zero, simplify to $x = a$.

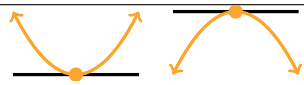
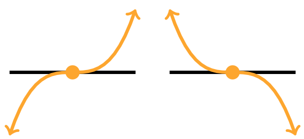
  For example, if $P(x) = (x - 5)(x - 5)(x + 2)$, then $x = 5$ has a multiplicity of 2 and $x = -2$ has a multiplicity of 1.

- When you have a repeated factor, normally you'll see it written more simply using exponents. In this case, the multiplicity is just the exponent of the factor that gives $x = a$.

  For example, if $P(x) = (x + 10)^4(x - 6)^3(x + 1)$, then $x = -10$ has a multiplicity of 4, $x = 6$ has a multiplicity of 3, and $x = -1$ has a multiplicity of 1.

Behavior Near Zeros

The multiplicity of a zero determines the behavior of the graph at that zero:

Multiplicity	Behavior
even	 The graph bounces off the x-axis at the zero.
odd	 The graph crosses through the x-axis at the zero.

5 Graphing Polynomial Functions

Goal

When we draw our graph, we want to have two features correct:

1. The end behavior
2. The behavior near the roots

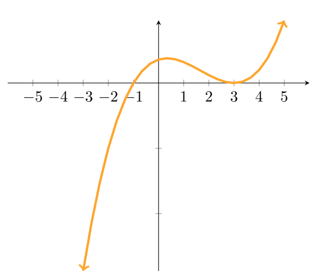
   Other details might be slightly incorrect, but we only need those two aspects right. For a better graph, you'd need to plug in a large number of points.

Method

1. Determine both the leading term and the factored form of the polynomial.
2. Use the factored form to find and plot the zeros.
3. Plug in and plot any extra points you want - y-intercept is nice.
4. Use the leading term to determine the end behavior. On the graph, draw the two sides onto the outer two most zeros.
5. Use the factored form to find the multiplicities of the zeros. Use these to fill in the "middle" part of the graph.

Examples

1. Graph $P(x) = (x+1)(2x^2 - 12x + 18)$



2. Graph $P(x) = -\frac{1}{4}x(x+1)^3(x-2)^2$