Section 3.3 Notes

Dividing Polynomials

1 Polynomial Long Division

Example of Long Division of Numbers
Let's compute $\frac{823}{5}$:

$$\begin{array}{r}
164 \\
5\overline{)823} \\
\underline{-5} \\
32 \\
\underline{-30} \\
23 \\
\underline{-20} \\
3
\end{array}$$

The quotient is 164 and the remainder is 3:

$$\boxed{\frac{823}{5} = 164 + \frac{3}{5}}$$

Rules

- The polynomials must always be written in descending order of exponents.

  $(x^3 - 2x^4 + 5x^2) \to (-2x^4 + x^3 + 5x^2)$

- If you have any missing terms, fill those terms in using zeros for the coefficient:

  $(x^4 - 3x^2 + 1) \to (x^4 + 0x^3 - 3x^2 + 0x + 1)$

- If the coefficients don't divide in evenly, you'll end up with fractional coefficients - that's completely fine.

Example of Long Division with Polynomials
Let's compute $\frac{x^2-3x+4}{x+1}$:

$$\begin{array}{r}
x - 4 \\
x+1\overline{)\ x^2 - 3x + 4} \\
\underline{-x^2 \quad - x} \\
-4x + 4 \\
\underline{4x + 4} \\
8
\end{array}$$

The quotient is $x-4$ and the remainder is 8:

$$\boxed{\frac{x^2 - 3x + 4}{x+1} = x - 4 + \frac{8}{x+1}}$$

Examples

1. Divide $8x^4 + 6x^2 - 3x + 1$ by $2x^2 - x + 2$

   $$\boxed{4x^2 + 2x + \frac{-7x+1}{2x^2 - x + 2}}$$

2. Compute $\dfrac{1+2x^2+3x^3}{2x-4}$

$$\boxed{\frac{3}{2}x^2+4x+8+\frac{33}{2x-4}}$$

2 Synthetic Division

Rules

- The polynomials must always be written in descending order of exponents.
- If you have any missing terms, fill those terms in using zeros for the coefficient.
- The divisor/denominator must have the form $x+c$ or $x-c$. If the divisor looks any different, you must go back to using long division.

Example of Synthetic Division

Let's compute $\frac{x^3-3x^2+1}{x-4}$:

$$\begin{array}{c|cccc} & 1 & -3 & 0 & 1 \\ 4 & & 4 & 4 & 16 \\ \hline & \underbrace{1 \quad 1 \quad 4}_{\text{Quotient}} & & & 17 \end{array}$$

Remainder

The quotient is x^2+x+4 and the remainder is 17:

$$\boxed{\frac{x^2-3x+4}{x+1} = x-4+\frac{8}{x+1}}$$

Examples

- Divide x^3-27 by $x-3$.

  $$\boxed{x^2+3x+9}$$

- Compute $\frac{x^4+2x^2+x-4}{x+2}$

  $$\boxed{x^3-2x^2+6x-11+\tfrac{18}{x+2}}$$

Forms of the Question

- So far, we've been asked to find $\frac{P(x)}{D(x)}$, which we've written as

  $$\frac{P(x)}{D(x)} = Q(x)+\frac{R(x)}{D(x)}$$

- However, some questions will give you $P(x)$ and $D(x)$, and then ask you to find $Q(x)$ and $R(x)$ where

  $$P(x) = D(x)\cdot Q(x)+R(x)$$

  You solve this *exactly* the same way - they're just multiplying the first equation through by $D(x)$ to solve for $P(x)$.

Examples
For each $P(x)$ and $D(x)$, use long division or synthetic division to find $Q(x)$ and $R(x)$ so that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

1. $P(x) = 4x^3 + 3x^2 - 2x + 5$ and $D(x) = x + 3$

   $\boxed{4x^3 + 3x^2 - 2x + 5 = (x + 3) \cdot (4x^2 - 9x + 25) - 70}$

2. $P(x) = x^4 - 2x^2 + x + 1$ and $D(x) = x^2 - x + 2$

   $\boxed{x^4 - 2x^2 + x + 1 = (x^2 - x + 2) \cdot (x^2 + x - 3) - 4x + 7}$

3 Theorems

Remainder Theorem

If $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

- This comes from when you write the problem solved for $P(x)$.
- Remember that when you divide by $x - c$, your remainder is always just a number, which we'll call r:

$$\begin{aligned} P(x) &= D(x) \cdot Q(x) + R(x) \\ P(x) &= (x - c) \cdot Q(x) + r \\ P(c) &= (c - c) \cdot Q(c) + r \text{ (Plug in } x = c.) \\ P(c) &= 0 \cdot Q(c) + r \\ P(c) &= r \end{aligned}$$

Example

Compute $P(-7)$ using the Remainder Theorem.

$$P(x) = 5x^4 + 30x^3 - 40x^2 - 36x + 14$$

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Factor Theorem

c is a zero of $P(x)$ (in other words, $P(c) = 0$) exactly when $x - c$ is a factor of $P(x)$

- To see why this makes sense, think of how we find zeros of a polynomial.
- For example, we'll find the zeros of $P(x) = 2x^2 + 5x - 3$

  $P(x) = (2x - 1)(x + 3)$

$0 = (2x - 1)(x + 3)$

$2x - 1 = 0$ or $x + 3 = 0$

$x = \frac{1}{2}$ or $x = -3$

- Notice how $\frac{1}{2}$ came from the factor $(2x - 1)$ and -3 came from the factor $(x + 3)$.
- It might look like this doesn't match up with the Factor Theorem, since the factors don't look like $x - c$, but we could have manipulated the factors a little bit more:

  $(2x - 1) = 2\left(x - \frac{1}{2}\right)$

  $(x + 3) = (x - (-3))$

Examples

- Show that $c = 3$ is a zero of $P(x) = x^3 - x^2 - 11x + 15$ and find the other zeros.

  3 is a zero since $(x - 3)$ is a factor.
  The other zeros are $x = -1 \pm \sqrt{6}$

- Find a polynomial with integer coefficients of degree 3 with zeros 2, $\frac{1}{2}$, and -1. Write the answer in the expanded form.

  $P(x) = 2x^3 - 3x^2 - 3x + 2$