Section 3.4 and 3.6 Notes

Real and Complex Zeros of Polynomials

1 Useful Theorems

Complete Factorization Theorem

If $P(x)$ is a polynomial with degree n, and real coefficients, then it can be *completely factored* into linear factors:

$$P(x) = a(x - c_1)(x - c_2) \dots (x - c_n)$$

- a is the leading coefficient when $P(x)$ is written in the expanded form.
- $c_1,\ c_2, \dots,\ c_n$ are the zeros of the polynomial (although the list might have repeats).
- $c_1,\ c_2, \dots,\ c_n$ might include complex numbers.
- If complex numbers are not allowed, you can factor $P(x)$ into a combination of linear and quadratic factors.

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Rational Roots Theorem

Suppose $P(x)$ is a polynomial with leading coefficient a and constant term b, and *all* its coefficients are integers. Every possible rational zero has the form

$$\pm \frac{\text{factor of } b}{\text{factor of } a}$$

2 Factoring Polynomials

Method

Let's factor $P(x) = x^3 + 3x^2 - 4$

1. List all possible rational zeros using the Rational Roots Theorem.

$$\begin{aligned} \pm\frac{\text{factor of } 4}{\text{factor of } 1} &= \pm\frac{1,2,4}{1} \\ &= \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1} \\ &= \pm 1, \pm 2, \pm 4 \end{aligned}$$

2. Go down the list, checking each possible zero using synthetic division until you get a remainder of 0.

$$\begin{array}{r|rrrr} & 1 & 3 & 0 & -4 \\ 1 & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array} \checkmark$$

$$P(x) = (x-1)(x^2+4x+4)$$

3. Keep repeating the last step on just the "leftover" quotient until the "leftover" part is a quadratic. Each time you successfully get a remainder of 0, try the same number again, because you can have repeated factors.

4. Once you just need to factor a quadratic, you can factor it normally or by using the quadratic formula.

$$\begin{aligned} P(x) &= (x-1)(x^2+4x+4) \\ P(x) &= (x-1)(x+2)(x+2) \\ P(x) &= (x-1)(x+2)^2 \end{aligned}$$

Examples

Completely factor and find the zeros of the following polynomials:

1. $P(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4$

$$P(x) = 2(x+1)(x-2)^2\left(x - \tfrac{1}{2}\right)$$
Zeros: $-1, 2, \frac{1}{2}$

2. $P(x) = x^4 - 7x^3 + 14x^2 - 3x - 9$

$$P(x) = (x-3)^2\left(x - \tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\right)\left(x - \tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right)$$
Zeros: $3, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

3. $P(x) = 2x^3 - 8x^2 + 9x - 9$

$$P(x) = 2(x-3)\left(x - \tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\mathrm{i}\right)\left(x - \tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\mathrm{i}\right)$$
Zeros: $3, \frac{1}{2} \pm \frac{\sqrt{5}}{2}\mathrm{i}$

3 Less Useful Theorems

Descartes' Rule of Signs
Suppose $P(x)$ is a polynomial with real number coefficients, written in descending order of exponents.

1. Count the number of variations in sign from one term to the next of $P(x)$. This number, or a number less than it by an even number tells you the number of positive zeros.
2. Count the number of variations in sign from one term to the next of $P(-x)$. This number, or a number less than it by an even number tells you the number of negative zeros.

As an example, look at $P(x) = x^3 - 6x^2 + 11x - 6$

$$P(x) = x^3 - 6x^2 + 11x - 6$$

$$\begin{aligned} x^3 &\to -6x^2 & &(+1 \text{ sign change}) \\ -6x^2 &\to 11x & &(+1 \text{ sign change}) \\ 11x &\to -6 & &(+1 \text{ sign change}) \end{aligned}$$

$P(x)$ has 3 sign changes. This means it has either 3 postive zeros or 1 positive zero.

As a note, it turns out the zeros are 1, 2 and 3.

$$\begin{aligned} P(-x) &= (-x)^3 - 6(-x)^2 + 11(-x) - 6 \\ &= -x^3 - 6x^2 - 11x - 6 \end{aligned}$$

$$\begin{aligned} -x^3 &\to -6x^2 & &(\text{no sign change}) \\ -6x^2 &\to -11x & &(\text{no sign change}) \\ -11x &\to -6 & &(\text{no sign change}) \end{aligned}$$

$P(-x)$ has 0 sign changes. This means it has no negative zeros.

The Upper and Lower Bound Theorem
Suppose a is a postive number.

1. If we divide $P(x)$ by $x - a$ using synthetic division and the answer row contains no negative numbers, then a is an uppoer bound for the real zeros of $P(x)$.

   If a is an *upper bound* for the zeros, then all the zeros need to be less than a (or equal).

2. If we divide $P(x)$ by $x + a$ using synthetic division and the answer row alternates back and forth from positive to negative, then $-a$ is a lower bound for the real zeros of $P(x)$.

   If a is a *lower bound* for the zeros, then all the zeros need to be greater than a (or equal).

As an example, look at $P(x) = x^3 - x^2 - 8x + 12$. We'll check $x = 4$ and $x = -4$.

$$\begin{array}{r|rrrr} & 1 & -1 & -8 & 12 \\ 4 & & 4 & 12 & 16 \\ \hline & 1 & 3 & 4 & 28 \end{array} \qquad \begin{array}{r|rrrr} & 1 & -1 & -8 & 12 \\ -4 & & -4 & 20 & -48 \\ \hline & 1 & -5 & 12 & -36 \end{array}$$

From this we see that 4 is an upper bound for all the zeros and -4 is a lower bound for all the zero. As a note, it turns out the zeros are -3 and 2.

Example

Factor $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$, making use of Descartes' Rule of Signs and the Upper and Lower Bound Theorem.

$$\boxed{P(x) = (x-3)^2(x-2\mathrm{i})(x+2\mathrm{i})}$$

Conjugate Zeros Theorem
Suppose $P(x)$ has real coefficients. If $a + b\mathrm{i}$ is a zero of $P(x)$, then $a - b\mathrm{i}$ is also a zero.

Example

Find a polynomial with integer coefficients and the roots $\frac{3}{2}$, -1, and $2 - \mathrm{i}$

$$\boxed{P(x) = 2x^4 - 9x^3 + 11x^2 + 7x - 15}$$