Section 3.5 Notes

Complex Numbers

1 Complex Numbers

Definitions

- In this section, we break one of our rules: thou shalt not take the square root of a negative number!
- To do this, mathematicians made up a new number called i:
  - $\mathrm{i} = \sqrt{-1}$
  - The essential property of i is that $\mathrm{i}^2 = -1$.
  - i is a number, but it is not a *Real Number* - it is sometimes called the "imaginary unit."
- A *complex number* has the form $\boxed{a + b\mathrm{i}}$, where a and b are real numbers.
  - a is called the *real part*
  - b is called the *imaginary part*

2 Arithmetic Operations

Addition and Subtraction

$$\boxed{\begin{gathered}(a+b\mathrm{i}) + (c+d\mathrm{i}) = (a+c) + (b+d)\mathrm{i} \\ (a+b\mathrm{i}) - (c+d\mathrm{i}) = (a-c) + (b-d)\mathrm{i}\end{gathered}}$$

- Add or subtract like terms
- Example 1: $(2+\mathrm{i}) + (4-3\mathrm{i}) = 6 - 2\mathrm{i}$
- Example 2: $(4-2\mathrm{i}) - (4-8\mathrm{i}) = 4 - 2\mathrm{i} - 4 + 8\mathrm{i} = 6\mathrm{i}$

Multiplication

$$\boxed{(a+b\mathrm{i})(c+d\mathrm{i}) = (ac-bd) + (ad+bc)\mathrm{i}}$$

- FOIL, and use the property that $\mathrm{i}^2 = -1$ to fully simplify.
- For example:

$$\begin{aligned}(2-3\mathrm{i})(1+4\mathrm{i}) &= 2 + 8\mathrm{i} - 3\mathrm{i} - 12\mathrm{i}^2 \\ &= 2 + 5\mathrm{i} - 12(-1) \\ &= 2 + 12 + 5\mathrm{i} \\ &= 14 + 5\mathrm{i}\end{aligned}$$

Division

$$\frac{a+b\mathrm{i}}{c+d\mathrm{i}} = \frac{ac+ad}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}$$

- When you are simplifying $\frac{a+b\mathrm{i}}{c+d\mathrm{i}}$, the goal is to get it back into the form of a normal complex number: $\boxed{} + \boxed{}\mathrm{i}$.
- These are really just "rationalize the denominator" problems in disguise - multiply the top and bottom by the conjugate of the bottom and simplify:

$$c+d\mathrm{i} \longleftrightarrow c-d\mathrm{i}$$

Examples

1. $\frac{2-3\mathrm{i}}{4+3\mathrm{i}}$

   $\boxed{\frac{5}{19} - \frac{18}{19}\mathrm{i}}$

2. $\frac{3\mathrm{i}}{2-7\mathrm{i}}$

   $\boxed{-\frac{21}{53} + \frac{6}{53}\mathrm{i}}$

3. $\frac{3+\mathrm{i}}{15\mathrm{i}}$

   $\boxed{\frac{1}{15} - \frac{1}{5}\mathrm{i}}$

3 Square Roots of Negative Numbers

Simplifying Square Roots with Negative Numbers

- Unfortunately, if a and b are *both* negative,

$$\sqrt{ab} \neq \sqrt{a}\sqrt{b}$$

- However, as long as just *one* of a or b is nonnegative, the property does hold:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

- Specifically, we can pull out $\sqrt{-1}$:

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2\mathrm{i}$$

Examples
Simplify.

1. $\sqrt{\frac{1}{4}}\sqrt{-64}$

   $\boxed{4\mathrm{i}}$

2. $\dfrac{\sqrt{-36}}{\sqrt{-4}\sqrt{-9}}$

$\boxed{-\mathrm{i}}$

4 Complex Roots of a Quadratic

Finding Complex Roots

- One of our methods for solving quadratic equations was by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In Section 1.5, we had the rule that if the inside the the root was negative, there was no solution. However, if we are allowed to have complex solutions (not just real number solutions), we actually can get an answer.
- Every quadratic with complex roots *must* be solved using the quadratic formula (or completing the square). It will *not* be factorable.

Examples
Find the zeros of the polynomial:

1. $P(x) = 3x^2 + 15$

   $\boxed{\pm\mathrm{i}\sqrt{5}}$

2. $P(x) = \frac{1}{2}x^2 - x + 5$

   $\boxed{1 \pm 3\mathrm{i}}$

3. $P(t) = t^2 + 3t + 3$

   $\boxed{\frac{3}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}}$