Section 4.3 Notes

Logarithmic Functions

1 Introduction

Definition

- $\log_a x$ answers the question "what exponent must a be raised to in order to get x?"
- In other words:
$$y = \log_a x \text{ means } a^y = x$$
- The *logarithmic function* with base a is given by the formula
$$f(x) = \log_a x$$
- Here a must be positive and $a \neq 1$.

Notation

- $\log_a x$ is also often written $\log_a(x)$.
- When written as $\log_a(x)$, this notation is meant to mirror $f(x)$ - function notation.
  - $\log_a$ is the name of the function
  - x is the input for the function
  - The parenthesis visually separate the name of the function from the input.
- When you speak about the function, it's okay to refer to "$\log_a$" by itself. However, when it's used in a formula or an equation, it must always have an input.
- For example
  - $y = 2 + \log_4(x)$ makes sense, whereas $y = 2 + \log_4$ does not.
  - To relate this back to function's we've dealt with before, $y = 2 + \sqrt[4]{x}$ makes sense, while $y = 2 + \sqrt[4]{}$ does not.

Examples
Calculate the following:

1. $\log_3 9$

   $\boxed{2}$

2. $\log_5 125$

   $\boxed{3}$

3. $\log_2 64$

   $\boxed{6}$

4. $\log_8 2$

   $\boxed{\frac{1}{3}}$

5. $\log_6 \frac{1}{36}$

   $\boxed{-2}$

2 Special Logarithms

Common Logarithm

$$\log x = \log_{10} x$$

This answers the question "what exponent must 10 be raised to in order to get x?"

Natural Logarithm

$$\ln x = \log_e x$$

This answers the question "what exponent must e (which is ≈ 2.7182818) be raised to in order to get x?"

Examples

Calculate the following:

1. $\log 1000$

   $\boxed{3}$

2. $\ln e^5$

   $\boxed{5}$

3. $\log \frac{1}{100}$

   $\boxed{-2}$

3 Properties

Basic Properties of Logarithms

	General Case	Common Log	Natural Log
1.	$\log_a 1 = 0$	$\log 1 = 0$	$\ln 1 = 0$
2.	$\log_a a = 1$	$\log 10 = 1$	$\ln e = 1$
3.	$\log_a a^x = x$	$\log 10^x = x$	$\ln e^x = x$
4.	$a^{\log_a x} = x$	$10^{\log x} = x$	$e^{\ln x} = x$

Explanation for the Properties

- For the first three properties, we can just transform something of the form $\log_a x = y$ into $a^y = x$.

- For the last property, we can transform something of the form $a^y = x$ into $\log_a x = y$.

	Rule	Reasoning
1.	$\log_a 1 = 0$	$a^0 = 1$
2.	$\log_a a = 1$	$a^1 = a$
3.	$\log_a a^x = x$	$a^x = a^x$
4.	$a^{\log_a x} = x$	$\log_a x = \log_a x$

Examples
Calculate the following using properties of logs:

1. $\log_4 64$

   $\boxed{3}$

2. $10^{\log 51}$

   $\boxed{51}$

3. $\log_3 \sqrt{27}$

   $\boxed{\frac{3}{2}}$

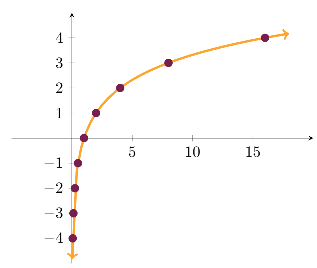
4. $\ln e^{200}$

   $\boxed{200}$

4 Logarithmic Graphs and Their Properties

Graphing Logarithmic Functions
Let's graph $y = \log_2 x$.

x	y
$\frac{1}{16}$	$\log_2\left(\frac{1}{16}\right) = \log_2\left(2^{-4}\right) = -4$
$\frac{1}{8}$	$\log_2\left(\frac{1}{8}\right) = \log_2\left(2^{-3}\right) = -3$
$\frac{1}{4}$	$\log_2\left(\frac{1}{4}\right) = \log_2\left(2^{-2}\right) = -2$
$\frac{1}{2}$	$\log_2\left(\frac{1}{2}\right) = \log_2\left(2^{-1}\right) = -1$
1	$\log_2(1) = 0$
2	$\log_2(2) = 1$
4	$\log_2(4) = \log_2\left(2^2\right) = 2$
8	$\log_2(8) = \log_2\left(2^3\right) = 3$
16	$\log_2(16) = \log_2\left(2^4\right) = 4$



Notice that the graph has a vertical asymptote along the y-axis.

Domain of Logarithmic Functions

In the last example, notice we didn't try to plug in $x = 0$ or any negative number.

Let's see what goes wrong:

$$y = \log_2(-8) \text{ means that } 2^y = -8$$

$\vdots$

$2^{-4} = \frac{1}{2^4} = \frac{1}{16} > 0$

$2^{-3} = \frac{1}{2^3} = \frac{1}{8} > 0$

$2^{-2} = \frac{1}{2^2} = \frac{1}{4} > 0$

$2^{-1} = \frac{1}{2^1} = \frac{1}{2} > 0$

$2^0 = 1 > 0$

$2^1 = 2 > 0$

$2^2 = 4 > 0$

$2^3 = 8 > 0$

$2^4 = 16 > 0$

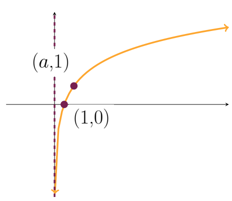
$\vdots$

- No matter what we plug in for y, every answer ends up positive!
- We *can't* plug a negative number or zero into a logarithm.
- The domain of $y = \log_2 x$ is $(0, \infty)$.

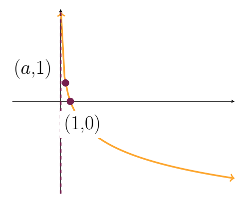
Rules for Finding Domain

1. If the expression contains $\boxed{\dfrac{A}{B}}$, then $\boxed{B \neq 0}$.
2. If the expression contains $\boxed{\sqrt{B}}$, then $\boxed{B \geq 0}$.
3. If the expression contains $\boxed{\log_a B}$, then $\boxed{B > 0}$.

General Logarithmic Graph
In general, there are just two basic shapes for $f(x) = \log_a x$:



$a > 1$



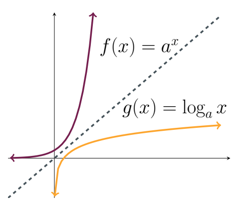
$a < 1$ (This is pretty uncommon)

- Again, they both have vertical asymptotes along the y-axis.
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$

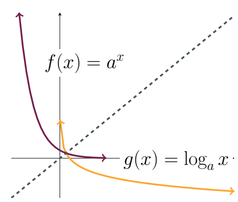
5 Relationship to Exponential Functions

Relationship through Graphs

- It turns out, $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions.
- If you look at their graphs, you can see they're reflections across the line $y = x$:



$a > 1$



$a < 1$

Relationship through Algebra

- You can also see that $f(x) = a^x$ and $g(x) = \log_a x$ are inverses by applying the Inverse Function Property:

$$\begin{aligned} f \circ g(x) &= a^{\log_a x} \\ &= x \text{ (from our fourth property of logarithms)} \end{aligned}$$

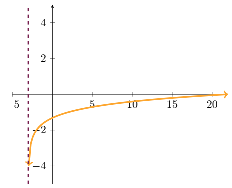
$$\begin{aligned} g \circ f(x) &= \log_a(a^x) \\ &= x \text{ (from our third property of logarithms)} \end{aligned}$$

6 Graphing Using Transformations

Examples

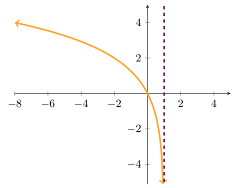
Graph the following using transformations and find their domains:

1. $y = \log_5(x+3) - 2$



Domain: $(-3, \infty)$

2. $y = 2\log_3(-x+1)$



Domain: $(-\infty, 1)$