Section 4.4 Notes

Laws of Logarithms

1 Laws of Logarithms

Formulas

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^z) = z\log_a x$

Justification

To see why these formulas work, you need to switch back into the exponential form and use our properties of exponents.

1. $\log_a(xy) = \log_a x + \log_a y$ uses the property $a^x a^y = a^{x+y}$.
   - Make up new variable names and switch to the exponential form:

$$r = \log_a x \rightarrow x = a^r$$
$$s = \log_a y \rightarrow y = a^s$$

   - Build back up to $\log_a(xy)$:

$$\begin{aligned} xy &= a^r a^s = a^{r+s} \\ \log_a(xy) &= \log_a(a^{r+s}) \\ \log_a(xy) &= r + s \text{ (One of our properties in 4.3)} \end{aligned}$$

   - Get rid of our made up variables:

$$\boxed{\log_a(xy) = \log_a x + \log_a y}$$

2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ uses the property $\frac{a^x}{a^y} = a^{x-y}$, but the steps are almost the same as the first one.

$$\begin{aligned} r &= \log_a x \rightarrow x = a^r \\ s &= \log_a y \rightarrow y = a^s \\ \frac{x}{y} &= \frac{a^r}{a^s} = a^{r-s} \\ \log_a\left(\frac{x}{y}\right) &= \log_a(a^{r-s}) \\ \log_a\left(\frac{x}{y}\right) &= r - s \end{aligned}$$

$$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$$

3. $\log_a(x^z) = z\log_a x$ uses the property $(a^x)^y = a^{xy}$, but the steps are almost the same as the first one.

$$\begin{aligned} r &= \log_a x \to x = a^r \\ x^z &= (a^r)^z = a^{rz} = a^{zr} \\ \log_a(x^z) &= \log_a(a^{zr}) \\ \log_a(x^z) &= zr \end{aligned}$$

$$\boxed{\log_a(x^z) = z\log_a x}$$

Examples
Use the Laws of Logarithms to expand the expressions:

1. $\log_2 \frac{2x^4}{y^2z^{10}}$

   $\boxed{1 + 4\log_2 x - 2\log_2 y - 10\log_2 z}$

2. $\log_4 \sqrt[3]{x^2 - 2x}$

   $\boxed{\frac{1}{3}\log_4 x + \frac{1}{3}\log_4(x-2)}$

3. $\log_5 \sqrt{\frac{3x}{y}}$

   $\boxed{\frac{1}{2}\log_5 3 + \frac{1}{2}\log_5 x - \frac{1}{2}\log_5 y}$

Use the Laws of Logarithms to combine the expressions:

4. $4\ln x - \frac{1}{2}\ln(x-1) + 2\ln 3$

   $\boxed{\ln \frac{9x^4}{\sqrt{x-1}}}$

5. $\log 4900 - 2\log 7$

   $\boxed{2}$

6. $4\log_5(x-4) + \log_5(x-1) - 3\log_5(x^2 - 5x + 4)$

   $\boxed{\log_5 \frac{x-4}{(x-1)^2}}$

2 Change of Base Formula

Formula

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

- The purpose of this formula is to allow you to switch to any new base you want (you can pick any positive number for b except for 1).
- Switching can allow you to plug things into a calculator (most calculators only have ln and log buttons).
- It can also give you another method of calculating "nice" logarithms,

Justification

- Make up a new variable name and switch to the exponential form:

$$r = \log_a x \to x = a^r$$

- Apply the $\log_b$ function to both sides:

$$\log_b x = \log_b(a^r)$$
$$\log_b x = r \log_b a$$

- Solve for r:

$$r = \frac{\log_b x}{\log_b a}$$

- Get rid of our made up variable:

$$\boxed{\log_a x = \tfrac{\log_b x}{\log_b a}}$$

Examples

1. $\log_9 27$

   $\boxed{\tfrac{3}{2}}$

2. $\log_8 128$

   $\boxed{\tfrac{7}{3}}$

3. $\log_5 37$

   $\boxed{2.243589\ldots}$