Section 4.5 Notes

Exponential and Logarithmic Equations

1 Exponential Equation

Definition

- An *exponential equation* is one where the variable appears in an exponent.
- There are many different categories that these equations can fall into.

The equation simplifies to $a^X = a^Y$

- Set $X = Y$ and finish solving.
- For example:

$$\begin{aligned} 5^x - 2 &= 23 \\ 5^x &= 25 \\ 5^x &= 5^2 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 3^{x^2} &= 9^{2x-2} \\ 3^{x^2} &= (3^2)^{2x-2} \\ 3^{x^2} &= 3^{4x-4} \\ x^2 &= 4x - 4 \\ x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

The equation simplifies to $a^X = b^Y$

- Apply your favorite type of log on both sides.
- Bring down the exponents using the third Law of Logarithms
- Finish solving.
- For example:

$$\begin{aligned} 4^{x+1} &= 7^{1-2x} \\ \ln 4^{x+1} &= \ln 7^{1-2x} \\ (x+1)\ln 4 &= (1-2x)\ln 7 \\ x\ln 4 + \ln 4 &= \ln 7 - 2x\ln 7 \\ x\ln 4 + 2x\ln 7 &= \ln 7 - \ln 4 \\ x(\ln 4 + 2\ln 7) &= \ln 7 - \ln 4 \\ x &= \frac{\ln 7 - \ln 4}{\ln 4 + 2\ln 7} \end{aligned}$$

Quadratic-Type Exponential Equations

- Use a substitution to make the equation a quadratic.
- Solve for the new variable.
- Go back to the original variable and finish solving.
- For example:

$$\begin{gathered}
e^{2x} - 4e^x - 5 = 0 \\
(e^x)^2 - 4e^x - 5 = 0 \\
\text{Make the substitution } y = e^x \\
y^2 - 4y - 5 = 0 \\
(y-5)(y+1) = 0 \\
y = 5 \text{ or } y = -1 \\
e^x = 5 \text{ or } e^x = -1 \\
x = \ln 5 \text{ or } x = \cancel{\ln(-1)}
\end{gathered}$$

The variable appears both in and out of exponents

- In general, you can't really solve these kinds of equations except numerically (with a computer).
- However, the problems from our class are designed to nicely factor.
- For example:

$$\begin{aligned}
2x^2e^{-x} + xe^{-x} &= 0 \\
xe^{-x}(2x+1) &= 0 \\
x = 0 \text{ or } e^{-x} = 0 \text{ or } 2x+1 &= 0 \\
x = 0 \text{ or } -x = \cancel{\ln 0} \text{ or } x &= -\frac{1}{2}
\end{aligned}$$

Examples

1. Find the time required for an investment of \$5000 to grow to \$8000 at an interest rate of 7.5%, compounded quarterly.

$$\boxed{t = \frac{\ln\left(\frac{8}{5}\right)}{4\ln(1.01875)} \approx 6.32528\ldots \text{ years}}$$

2. How long will it take an investment of \$3000 to double if the interest rate is 6.5% per year, compounded continuously?

$$\boxed{t = \frac{\ln 2}{0.065} \approx 10.6638\ldots \text{ years}}$$

2 Logarithmic Equations

Definition

- A *logarithmic equation* is one where the variable appears inside a logarithm.
- Although there are many possible types of these, we will only solving ones with a specific form.

Method

1. If there are any constant terms, move all those to one side and the log terms to the other side.
2. Combine all the log terms on each side into a single logarithm using the Laws of Logarithms.
3. At this point, it should be in one of two forms:
   - $\log_a X = Y$. Rewrite as $a^Y = X$ and finish solving.
   - $\log_a X = \log_a Y$. Set $X = Y$ and finish solving.
4. You need to check your answers against the domain of the original problem.

1. $\log(4 - x) = 2$

   $\boxed{x = -96}$

2. $\log_3(1 - x) = 1 - \log_3(-x - 1)$

   $\boxed{x = -2}$

3. $\ln x + \ln(x - 1) = \ln 20$

   $\boxed{x = 5}$