Section 4.6 Notes

Modeling with Exponential and Logarithmic Equations

Exponential Growth
A population that experiences exponential growth will increase according to the equation

$$n(t) = n_0 e^{rt}$$

where

- t is the time (in any given units)
- $n(t)$ is the population at time t
- n_0 is the initial population size
- r is the relative growth rate.

A population that experiences exponential growth also has a corresponding doubling time. If the doubling time is a, then the population will increase according to the equation

$$n(t) = n_0 e^{rt}, \text{ where } r = \frac{\ln 2}{a}$$

You can also simplify this equation to

$$n(t) = n_0 2^{t/a}$$

Example
The bat population in a certain Midwestern county was 350,000 in 2009 and the observed doubling time for the population is 25 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2009.

$$\boxed{n(t) = 350000 \cdot 2^{t/25}}$$

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2009.

$$\boxed{n(t) = 350000 e^{\frac{\ln 2}{25} t} \text{ or approximately } n(t) = 350000 e^{0.0277t}}$$

(c) Estimate when the population will reach 2 million.

We can use either of the two formulas for this problem, so pick your favorite.

(1) $2000000 = 350000 \cdot 2^{t/25}$

(2) $\dfrac{2000000}{350000} = \dfrac{\cancel{350000} \cdot 2^{t/25}}{\cancel{350000}}$

(3) $\dfrac{40}{7} = 2^{t/25}$

(4) $\ln \dfrac{40}{7} = \ln 2^{t/25}$

(5) $\ln \dfrac{40}{7} = \dfrac{t}{25} \ln 2$

(6) $\dfrac{25}{\ln 2} \cdot \ln \dfrac{40}{7} = \dfrac{t \cancel{\ln 2}}{\cancel{25}} \cdot \dfrac{\cancel{25}}{\cancel{\ln 2}}$

(7) $t = \dfrac{25 \ln\left(\frac{40}{7}\right)}{\ln 2} \approx 63$ years

$\boxed{\text{The population will reach 2 million by the year 2072.}}$

Exponential Decay

A sample that experiences exponential decay will decrease according to the equation

$$m(t) = m_0 e^{-rt}$$

where

- t is the time (in any given units)
- $m(t)$ is the sample size at time t
- m_0 is the initial sample size
- r is the relative decay rate.

A sample that experiences exponential decay also has a corresponding half-life. If the half-life if h, then the sample will decrease according to the equation

$$m(t) = m_0 e^{-rt}, \text{ where } r = \frac{\ln 2}{h}$$

You can also simplify this equation to

$$m(t) = m_0 2^{-t/h}$$

Example

The half-life of radium-226 is 1600 years. Suppose we have a 22 mg sample.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

$\boxed{m(t) = 22 \cdot 2^{-t/1600}}$

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

$\boxed{n(t) = 350000e^{-\frac{\ln 2}{1600}t} \text{ or approximately } n(t) = 350000e^{-0.0004332t}}$

(c) How much of the sample will remain after 4000 years?

$$\begin{aligned} m(4000) &= 22 \cdot 2^{-4000/1600} \\ &= 22 \cdot 2^{-2.5} \\ &\approx 3.8891 \end{aligned}$$

$\boxed{3.9 \text{ mg}}$

(d) After how long will only 18 mg of the sample remain?

(1) $18 = 22 \cdot 2^{-t/1600}$

(2) $\dfrac{18}{22} = \dfrac{\cancel{22} \cdot 2^{-t/1600}}{\cancel{22}}$

(3) $\dfrac{9}{11} = 2^{-t/1600}$

(4) $\ln \dfrac{9}{11} = \ln 2^{-t/1600}$

(5) $\ln \dfrac{9}{11} = -\dfrac{t}{1600} \ln 2$

(6) $\dfrac{-1600}{\ln 2} \cdot \ln \dfrac{9}{11} = -\dfrac{t \cancel{\ln 2}}{\cancel{1600}} \cdot \dfrac{\cancel{-1600}}{\cancel{\ln 2}}$

(7) $t = -\dfrac{1600 \ln\left(\frac{9}{11}\right)}{\ln 2} \approx 463$ years

It will take 463 years before only 18 mg remain.

Newton's Law of Cooling

An object that's hotter/colder than it's surrounding environment will cool off/heat up according to the equation

$$T(t) = T_S + (T_0 - T_S)e^{-kt}$$

where

- t is the time (in any given units)
- $T(t)$ is temperature of the object at time t
- T_S is temperature of the surrounding environment
- T_0 is the initial temperature of the object
- k is a constant that depends on the physical properties of the object and its surrounding. It will change based on how easily they transfer heat to each other.

Example

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature is 150° in half an hour, what is the temperature after 45 minutes?

I'll be doing the problem in minutes instead of hours, so 30 min. First we need to find k:

(1) $150 = 75 + (185 - 75)e^{-k(30)}$

(2) $75 = 110e^{-30k}$

(3) $\dfrac{75}{110} = \dfrac{\cancel{110}e^{-30k}}{\cancel{110}}$

(4) $\dfrac{15}{22} = e^{-30k}$

(5) $\ln \dfrac{15}{22} = \ln e^{-30k}$

(6) $\ln \dfrac{15}{22} = -30k$

(7) $k = -\dfrac{\ln\left(\frac{15}{22}\right)}{30} \approx 0.01277$

Example

Next we plug back in to find the temperature:

$$\begin{aligned} T(45) &\approx 75 + (185 - 75)e^{-0.01277(45)} \\ &\approx 75 + 110e^{-0.5745} \\ &\approx 75 + 110(0.5630) \\ &\approx 136.9292 \end{aligned}$$

137°F

Example

(b) When will the turkey cool to 100°F?

(1) $100 = 75 + 110e^{-0.01277t}$

(2) $25 = 110e^{-0.01277t}$

(3) $\dfrac{25}{110} = \dfrac{\cancel{110}e^{-0.01277t}}{\cancel{110}}$

(4) $\dfrac{5}{22} = e^{-0.01277t}$

(5) $\ln \dfrac{5}{22} = \ln e^{-0.01277t}$

(6) $\ln \dfrac{5}{22} = -0.01277t$

(7) $t = -\dfrac{\ln\left(\frac{5}{22}\right)}{0.01277} \approx 116.0549$

116 minutes or 1 hour and 56 minutes

Richter Scale

The magnitude of an earthquake is given by

$$M = \log \frac{I}{S}$$

where

- M is the magnitude of the earthquake (in the Richter Scale)
- I is the intensity of the earthquake (this is the length of the squiggly lines that are drawn by the seismometer)
- S is the intensity of a "standard" earthquake, specifically $S = 1$ micron $= 10^{-4}$ cm.

Example

The Alaska earthquake of 1964 had a magnitude of 8.6 on the Richter Scale. How many times more intense was this than the 1906 San Francisco earthquake (which had a magnitude of 8.3 on the Richter Scale)?

- I_A: the intensity of the Alaska earthquake
- I_{SF}: the intensity of the San Francisco earthquake

(1) Let's solve for I:

$$M = \log \frac{I}{S}$$
$$10^M = \frac{I}{S}$$
$$I = S \cdot 10^M$$

(2) Plug in our info:

$$I_A = S \cdot 10^{8.6}$$
$$I_{SF} = S \cdot 10^{8.3}$$

(3) And find the ratio:

$$\frac{I_A}{I_{SF}} = \frac{\cancel{S} \cdot 10^{8.6}}{\cancel{S} \cdot 10^{8.3}} = 10^{8.6-8.3} = 10^{0.3} \approx 1.9953$$

The Alaska earthquake was about 2 times more intense.

pH Scale

The pH (or measure of acidity/alkalinity) of a solution is given by

$$\text{pH} = -\log[\text{H}^+]$$

where

- pH is the pH of the solution (less than 7 is acidic, 7 is neutral, and greater than 7 is alkaline/basic)
- $[\mathrm{H}^+]$ is the concentration of free hydrogen ions in the solution measured in moles/liter.

Although it's uncommon in our class, for this formula both pH and $[\mathrm{H}^+]$ are each treated as single variables, despite being written using multiple letters/symbols.

Example
The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.

(a) Vinegar: pH=3.0

(b) Milk: pH=7.3

Solve for $[\mathrm{H}^+]$:

$$\begin{aligned} \mathrm{pH} &= -\log[\mathrm{H}^+] \\ -\mathrm{pH} &= \log[\mathrm{H}^+] \\ [\mathrm{H}^+] &= 10^{-\mathrm{pH}} \end{aligned}$$

(a)

$$[\mathrm{H}^+] = 10^{-3}$$

$$\boxed{1.0 \times 10^{-3}}$$

(b)

$$\begin{aligned} [\mathrm{H}^+] &= 10^{-7.3} \\ &\approx 5.01187 \times 10^{-8} \end{aligned}$$

$$\boxed{5.0 \times 10^{-8}}$$

Decibel Scale
The intensity of a sound measured in decibels (dB) is given by

$$B = 10 \log \frac{I}{I_0}$$

where

- B is the intensity of the sound measured in decibels
- I is the intensity of the sound measured in Watts/meter^2
- I_0 is a "reference" intensity. Specifically, $I_0 = 10^{-12}$ Watts/meter^2.

Example
The noise from a power mower was measured at 106 dB. The noise level at a rock concert was measured at 120 dB. Find the ratio of the intensity of the rock music to that of the power mower.

- I_{PM}: intensity of the power mower
- I_{RC}: intensity of the rock concert

(1) Let's solve for I:

$$\begin{aligned} \frac{B}{10} &= \log \frac{I}{I_0} \\ \frac{I}{I_0} &= 10^{B/10} \\ I &= I_0 \cdot 10^{B/10} \end{aligned}$$

(2) Plug in our info:

$$\begin{aligned} I_{PM} &= I_0 \cdot 10^{106/10} \\ &= I_0 \cdot 10^{10.6} \\ I_{RC} &= I_0 \cdot 10^{120/10} \\ &= I_0 \cdot 10^{12} \end{aligned}$$

(3) And find the ratio:

$$\begin{aligned} \frac{I_{RC}}{I_{PM}} &= \frac{\cancel{I_0} \cdot 10^{12}}{\cancel{I_0} \cdot 10^{10.6}} \\ &= 10^{12-10.6} = 10^{1.4} \\ &\approx 25.1189 \end{aligned}$$

$\boxed{\text{The rock music was about 25 times more intense than the mower.}}$